

SYMMETRIC BILINEAR FORMS AND VERTICES IN CHARACTERISTIC 2

JOHN C. MURRAY

In memory of J. A. Green

ABSTRACT. Let G be a finite group and let k be an algebraically closed field of characteristic 2. Suppose that M is an indecomposable kG -module which affords a non-degenerate G -invariant symmetric bilinear form. We assign to M a collection of 2-subgroups of G called its symmetric vertices, each of which contains a Green vertex of M with index at most 2. If M is irreducible then its symmetric vertices are uniquely determined, up to G -conjugacy.

If B is the real 2-block of G containing M , we show that each symmetric vertex of M is contained in an extended defect group of B . Moreover, we characterise the extended defect groups in terms of symmetric vertices.

In order to prove these results, we develop the theory of involutory G -algebras. This allows us to translate questions about symmetric kG -modules into questions about projective modules of quadratic type.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let G be a finite group and let k be an algebraically closed field of characteristic 2. By a G -form on a kG -module M we mean a non-degenerate G -invariant symmetric or symplectic bilinear form $B : M \times M \rightarrow k$. We then refer to (M, B) as a symmetric or symplectic kG -module and say that M has symmetric or symplectic type. We say that M has quadratic type if it affords a symplectic G -form that is the polarization of a G -invariant quadratic form. Note that each symplectic form is symmetric, as $\text{char}(k) = 2$, and each indecomposable kG -module of symmetric (or symplectic) type belongs to a real 2-block of G as it is self-dual.

W. Willems noted that the Krull-Schmidt theorem fails for symmetric kG -modules [30, 3.13]. This paper is our attempt to understand induction and restriction of symmetric modules in the absence of this fundamental tool. Our main idea is to use the Puig correspondence theorem [28, 19.1] for involutory G -algebras [18] to reduce the study of symmetric modules to symmetric projective modules. The latter can be analysed using the methods of [8].

Recall that M is H -projective, for $H \leq G$, if M is a component of an induced module $\text{Ind}_H^G L$, for some kH -module L . If M is indecomposable, a vertex of M is a subgroup V of G which is minimal subject to M being V -projective. A V -source of M is then a kV -module Z such that M is a component of $\text{Ind}_V^G Z$. J. A. Green [10] proved that V is unique up to G -conjugacy, and Z up to $N_G(V)$ -conjugacy.

Date: December 11, 2015, Revisions March 14, 2016.

1991 Mathematics Subject Classification. Primary 20C20.

Key words and phrases. modular representations of finite groups, characteristic 2, symmetric and symplectic bilinear forms, G -algebras with involution, vertices and sources, block theory.

We extend the notion of H -projectivity to symmetric modules, saying that (M, B) or B is H -projective if (M, B) is a component of an induced symmetric module $\text{Ind}_H^G(L, B_L)$, for some symmetric kH -module (L, B_L) . Here $\text{Ind}_H^G(L, B_L)$ is $\text{Ind}_H^G L$ endowed with the standard induced G -form B_L^G . We say that M is symmetrically H -projective if it affords a H -projective symmetric form. If M is indecomposable we define a *symmetric vertex* of M to be a subgroup T of G that is minimal subject to M being symmetrically T -projective.

The standard proof of the uniqueness of Green-vertices uses Mackey's formula for the restriction of induced modules and the Krull-Schmidt theorem. An alternative uses Mackey's formula for products of relative traces in endomorphism rings. Both approaches fail for symmetric vertices, the latter because the product of two self-adjoint endomorphisms is not self-adjoint. Indeed, it is not hard to find a module which has two conjugacy classes of symmetric vertices:

Example 1.1. *The dihedral group D_{12} has two subgroups H_1 and H_2 that are isomorphic to S_3 . Let T_i be a Sylow 2-subgroup of H_i and let M_i be the unique non-trivial irreducible kH_i -module, for $i = 1, 2$. Then $\text{Ind}_{H_1}^{D_{12}} M_1 = \text{Ind}_{H_2}^{D_{12}} M_2$ is a projective indecomposable kD_{12} -module P . It is easy to see that T_1 and T_2 are symmetric vertices of P , but T_1 is not conjugate to T_2 in D_{12} .*

On the other hand, symmetric vertices are closely related to Green vertices:

Theorem 1.2. *Let M be an indecomposable kG -module which has symmetric type and let T be a symmetric vertex of M . Then T contains a Green vertex V of M and $[T : V] \leq 2$. More precisely, exactly one of the following is true:*

- (i) *M has a G -form which is non-degenerate on a component of $\text{Res}_V^G M$ that is a source of M . Then $T = V$ and M is in the principal 2-block of G .*
- (ii) *The sources of M are self-dual but each G -form on M is degenerate on any component of $\text{Res}_V^G M$ that is a source of M . Then $[T : V] = 2$.*
- (iii) *The sources of M are not self-dual. Then $[T : V] = 2$.*

In particular the symmetric vertices of an indecomposable kG -module are 2-subgroups of G . As regards proofs, we show that $[T : V] \leq 2$ in Lemma 4.6, and treat (i) in Proposition 5.13 and Corollary 5.14, (ii) in Proposition 5.15 and (iii) in Proposition 5.16. We give examples of all three cases in Section 5.8.

A given indecomposable module can have infinitely many non-isometric forms. Moreover a generic form on a module need not be projective relative to any symmetric vertex of the module. However the forms induced from a symmetric vertex have the following property:

Theorem 1.3. *Let M be an indecomposable kG -module which has symmetric type, let T be a symmetric vertex of M and let B be a T -projective G -form on M . Then B is H -projective, for $H \leq G$, if and only if $T \leq_G H$.*

In particular if M has a unique form, up to isometry, then it has a unique symmetric vertex, up to G -conjugacy. We prove this theorem in Lemma 5.17. In Example 1.1, each T_1 -projective form on P is not T_2 -projective, and vice-versa.

The trivial kG -module is of symmetric type and its symmetric vertices are just the Sylow 2-subgroups of G . P. Fong [4] noted that each non-trivial self-dual irreducible kG -module has a unique symmetric G -form, up to scalars, and this form is symplectic. So Theorem 1.3 implies:

Theorem 1.4. *The symmetric vertices of a self-dual irreducible kG -module are uniquely determined up to G -conjugacy.*

We now discuss applications to real 2-blocks. In fact this paper was motivated by our attempt to find an analogue of the extended defect group of a real 2-block for self-dual indecomposable kG -modules.

Recall that kG is a $k(G \times G)$ -module. Let \mathcal{B} be an indecomposable component of this module. Then \mathcal{B} is a block of kG , or a 2-block of G . R. Brauer defined a defect group of \mathcal{B} to be a Sylow 2-subgroup D of the centralizer of a certain element of G . J. A. Green showed that the diagonal subgroup $\Delta D \leq G \times G$ is a vertex of \mathcal{B} , as $k(G \times G)$ -module.

Now $g \rightarrow g^{-1}$, for $g \in G$, extends to an involutory k -algebra anti-automorphism $^\circ$ of kG called the contragredient map. Assume that \mathcal{B} is invariant under $^\circ$. Then \mathcal{B} is a real 2-block of G . R. Gow defined an extended defect group of \mathcal{B} to be a Sylow 2-subgroup E of the extended centralizer of a certain real element of G [6]. If \mathcal{B} is the principal block $E = D$ is a Sylow 2-subgroup of G . Otherwise $D \leq E$ and $[E : D] = 2$.

In [21] we used E to determine a vertex of \mathcal{B} , regarded as a module for the wreath product $G \wr C_2$ of G with a cyclic group of order 2. Now let B_1 be the $G \times G$ -form on kG with respect to which the elements of G form an orthonormal basis.

Theorem 1.5. *Let \mathcal{B} be a real 2-block of G , with extended defect group E . Then*

- (i) *Each symmetric vertex of an indecomposable \mathcal{B} -module is G -conjugate to a subgroup of E .*
- (ii) *Some self-dual irreducible \mathcal{B} -module has symmetric vertex E .*
- (iii) *The restriction of B_1 to \mathcal{B} is non-degenerate and ΔE -projective. So ΔE is a symmetric vertex of \mathcal{B} , regarded as a $G \times G$ -module.*

We prove (i) in Lemma 5.19, (ii) in Corollary 5.20 and (iii) in Lemma 5.21. Note that (i) and (ii) give a new proof of Gow's result [6, 2.2] that the extended defect groups of a real 2-block of G are uniquely determined up to G -conjugacy. However we do not know if all symmetric vertices of \mathcal{B} are $G \times G$ -conjugate to ΔE .

Regarding the rest of the paper, we give examples of the failure of the Krull-Schmidt Theorem for symmetric kG -modules in 2.1. The rest of Section 2 contains basic results on bilinear forms and involutory k -algebras. Lemma 2.3 is a lifting result for self-adjoint idempotents which is a mild generalization of [18, 1.4]. We shall use it extensively in the rest of the paper.

We consider adjoints, endomorphisms and bilinear forms in Section 3. Lemma 3.4 gives a bijection between G -endomorphisms and perfect G -pairings on pairs of submodules. Lemma 3.5 shows that self adjoint idempotents correspond to orthogonal direct summands. In Lemma 3.8 we give a new proof of [9, Proposition] on decompositions of symmetric modules. Finally Lemma 3.9 shows that in the presence of a G -involution each projective representation lifts to a representation.

In Section 4 we undertake a detailed exploration of form induction. This notion has appeared in many places, for example [9], [25] and [26]. D. Higman [13] provided a criterion for H -projectivity using the relative trace map. We define the notion of H -projectivity for symmetric forms in 4.1 and prove a symmetric version of Higman's Criterion in Lemma 4.3.

The deepest results of this paper are contained in Section 5. In 5.1 we consider symmetric bilinear forms on projective kG -modules. These can be studied in some

detail because of previous work such as [8] and [22]. For example each projective indecomposable module which has symplectic type is of quadratic type. We give a new criterion for this to occur in Proposition 5.8, simplified here as:

Theorem 1.6. *Let M be a non-trivial self-dual irreducible kG -module and let B be a symplectic G -form on M . Then $P(M)$ has quadratic type if and only if there is an involution $t \in G$ such that $B(tm, m) \neq 0$, for some $m \in M$.*

We interpret this result as follows. Suppose that $P(M)$ is not of quadratic type and let $t \in G$ be an involution. Then the image of t in $E(M)$ is an *alternate* linear transformation, in the terminology of [7]. Equivalently $M = M_1 \dot{+} M_2$, where M_1 and M_2 are isomorphic $k\langle t \rangle$ -modules that are totally isotropic with respect to B .

We have already indicated that the proof of Theorem 1.2 is scattered among Sections 4.3, 5.3, 5.4, and 5.5, and proofs of Theorems 1.3 and 1.4 are given in Section 5.6. We would like to highlight the following version of Proposition 5.12:

Theorem 1.7. *Let V be a 2-subgroup of G and let Z be an indecomposable kV -module with vertex V which is of symmetric type. Then among the indecomposable components of $\text{Ind}_V^G Z$ which have vertex V , there is a distinguished kG -module M characterised by any one of:*

- (a) *M has multiplicity one as a component of $\text{Ind}_V^G Z$.*
- (b) *M has odd multiplicity as a component of $\text{Ind}_V^G Z$.*
- (c) *Z is a B -component of $\text{Res}_V^G M$, for some G -form B on M .*
- (d) *Each V -projective form on $\text{Ind}_V^G Z$ is non-degenerate on M .*

If Z is the trivial kV -module, then M is just the Scott-module with vertex V . So this theorem extends the notion of a Scott-module from permutation modules to a much larger class of modules in characteristic 2. Like a Scott module, M will belong to the principal 2-block of G , but unlike a Scott module, M need not have a trivial submodule nor a trivial quotient module.

Unless stated otherwise, k is an algebraically closed field of characteristic 2. We relax the hypotheses on k in some sections, as will be indicated. Everywhere G is a finite group of even order. All vector spaces and algebras are finite dimensional and all kG -modules are left kG -modules. If M is a k -vector space then $E(M)$ is its endomorphism ring, and if M is a kG -module then $P(M)$ is its projective cover.

2. GENERALITIES

Throughout this section k is a field and M is a k -vector space.

2.1. Failure of the Krull-Schmidt theorem. This short subsection demonstrates one of the difficulties we face when dealing with symmetric bilinear forms in characteristic 2: certainly each symmetric kG -module decomposes as an orthogonal sum of indecomposables. However the symmetric modules occurring in such a sum are not uniquely determined up to isometry. In order to get to the point quickly, we defer defining all terms until later in the paper.

If $\text{char}(k) \neq 2$, R. Gow observed that each self-dual indecomposable kG -module has either symmetric or symplectic type. In [30, 3.5] W. Willems showed that such a module has a unique symmetric or symplectic form, up to isometry. Moreover he proved an analogue of the Krull-Schmidt theorem in [30, 3.11]: the decomposition of a symmetric or symplectic kG -module into orthogonal indecomposables is unique up to isometry.

For the rest of 2.1 we suppose that $\text{char}(k) = 2$. Then if M is an indecomposable kG -module it is easy to verify that its paired module $(M^* \oplus M, P)$ (defined in 3.2 below) is orthogonally indecomposable. We use this to construct an example that shows that the Krull-Schmidt theorem fails for symmetric kG -modules:

Example 2.1. *Let M be an indecomposable kG -module and let B be a G -form on M . Then the diagonal submodule of $(M, B) \perp (M, B) \perp (M, B)$ is isomorphic to (M, B) and its orthogonal complement is isomorphic to $(M^* \oplus M, P)$. So*

$$(M, B) \perp (M, B) \perp (M, B) \cong (M, B) \perp (M^* \oplus M, P),$$

yet even the number of indecomposable components is different on both sides.

We do not give proofs of these assertions, except to note that the two decompositions exist when G is trivial and $M \cong k$. The general case then follows easily from Lemma 2.3 applied in the context of Lemma 3.8.

Now let B_1 be the standard G -form on the regular module kG . For $x \in kG$, set $B_x(y, z) := B_1(yx, z)$, for all $y, z \in kG$. Then B_x is symplectic if and only if $B_1(x, 1) = 0$, and non-degenerate if and only if $B_1(x, x) \neq 0_k$. See Section 5.1 for details. We use these facts in our second example:

Example 2.2. [30, 3.13] *Let $V_4 = \{1, r, s, t\}$ be the Klein 4-group, and let M be the regular kV_4 -module. Consider the symplectic kV_4 -module $(M, B_r) \perp (M, B_s)$. Then for each $\alpha, \beta \in k$ with $\alpha \neq \beta$ we have*

$$(M, B_r) \perp (M, B_s) = (M, B_{\alpha r + \beta s}) \perp (M, B_{\beta r + \alpha s}).$$

But for units $x, y \in kV_4$, $(M, B_x) \cong (M, B_y)$ if and only if $y = \lambda x$ for some $\lambda \in k^\times$. So $(M, B_r) \perp (M, B_s)$ has an infinite number of orthogonal decompositions, no two of which have a common indecomposable component, even up to isometry.

2.2. Involutary algebras and idempotent lifting. Let A be a k -algebra such that $A/J(A)$ is split semi-simple. An involution on A is a k -linear automorphism $a \rightarrow a^\tau$ of A such that $\tau^2 = \text{id}_A$ and $(ab)^\tau = b^\tau a^\tau$ for all $a, b \in A$. Note in particular that τ fixes each scalar multiple of 1_A . Following Landrock and Manz [18] we call (A, τ) an involutory k -algebra.

Recall that A is a G -algebra if there is a homomorphism $G \rightarrow \text{Aut}(A)$, written $a \rightarrow {}^g a$ for $g \in G$ and $a \in A$. If in addition τ is an involution on A which commutes with the G -action, we call (A, τ) an involutory G -algebra. If $H \leq G$, we use A^H to denote the subalgebra of H -fixed points in A .

We prove a mild generalization of an idempotent lifting result [18, Lemma 1.4]. This will be used frequently, often without explicit reference, later in the paper. If I is a 2-sided ideal of A , set $\bar{A} = A/I$ and $\bar{a} = a + I$ in \bar{A} . This notation is ambiguous, but should be clear from the context.

Lemma 2.3. *Let (A, τ) be an involutory k -algebra, let I be a τ -invariant 2-sided ideal of A and let $a \in A \setminus I$.*

- (i) *If a is a primitive τ -invariant idempotent in A , then \bar{a} is a primitive τ -invariant idempotent in A/I .*
- (ii) *If \bar{a} is a τ -invariant idempotent in \bar{A} then there is a τ -invariant idempotent $e \in aAa^\tau$ with $\bar{e} = \bar{a}$ and $e = p(aa^\tau)$ for some polynomial p with $p(0) = 0$.*
- (iii) *If \bar{a} is a τ -invariant primitive idempotent in \bar{A} , then there is a τ -invariant primitive idempotent f in A such that $\bar{f} = \bar{a}$.*

Proof. Note that (\overline{A}, τ) is an involutory k -algebra, via $\overline{u}^\tau := \overline{u^\tau}$, for all $u \in A$.

Assume the hypothesis (i). Then [15, (2.3)] implies that \overline{a} is a primitive idempotent in A/I , and it is clear that $\overline{a}^\tau = \overline{a}$.

For (ii), we may assume that $\overline{1}$ and \overline{a} are linearly independent in \overline{A} . Set $b = aa^\tau$. Then $b^\tau = b$ and $\overline{b} = \overline{a}\overline{a}^\tau = \overline{a}$. We apply idempotent lifting [15, (3.2)] to the k -algebra $k[b]$ modulo its ideal $k[b] \cap I$. So there is an idempotent $e \in k[b]$ such that $\overline{e} = \overline{b}$. Then e is τ -invariant as b is τ -invariant and $k[b]$ is commutative. Write $e = p(b)$, where $p \in k[x]$. Then $\overline{e} = p(0)\overline{1} + (p(1) - p(0))\overline{a}$. So $p(0) = 0$.

Finally, assume (iii). Then by [15, (3.10)] there is a primitive idempotent $c \in A$ such that $\overline{c} = \overline{a}$. Applying (ii) to c , there is a τ -invariant idempotent $f \in cAc^\tau$ with $\overline{f} = \overline{c} = \overline{a}$. Then f is primitive in A as $cuc^\tau \rightarrow cuc^\tau c$ defines a k -algebra isomorphism of cAc^τ with the local algebra cAc . \square

We establish notation for points and multiplicity modules. Let A^\times be the group of multiplicative units in A . A point of A is an A^\times -conjugacy class ϵ of primitive idempotents of A . There is a unique maximal 2-sided ideal \mathfrak{M}_ϵ of A which does not contain any idempotent in ϵ . The multiplicity module of ϵ is the unique irreducible A -module P_ϵ annihilated by \mathfrak{M}_ϵ and the multiplicity algebra of ϵ is the simple quotient algebra A/\mathfrak{M}_ϵ . Notice that A/\mathfrak{M}_ϵ is isomorphic to the endomorphism ring $E(P_\epsilon)$ of P_ϵ . We identify the projection $\pi_\epsilon : A \rightarrow A/\mathfrak{M}_\epsilon$ with a map $A \rightarrow E(P_\epsilon)$.

Now ϵ^τ is also a point of A and $\mathfrak{M}_\epsilon \cap \mathfrak{M}_{\epsilon^\tau}$ is a τ -invariant ideal of A . Suppose that $\epsilon^\tau \neq \epsilon$. Then $A/\mathfrak{M}_\epsilon \cap \mathfrak{M}_{\epsilon^\tau} \cong A/\mathfrak{M}_\epsilon \times A/\mathfrak{M}_{\epsilon^\tau}$ is a semi-simple k -algebra with involution τ interchanging the two simple components. We identify the projection $\pi_{\epsilon, \epsilon^\tau} = \pi_\epsilon \times \pi_{\epsilon^\tau}$ with a map $A \rightarrow E(P_\epsilon) \times E(P_{\epsilon^\tau})$.

2.3. Symmetric and symplectic forms. A bilinear form on M is a map $B : M \times M \rightarrow k$ which is linear in each variable. So $m \rightarrow B(m, -)$ and $m \rightarrow B(-, m)$ are k -linear maps from M to its linear dual $M^* = \text{Hom}_k(M, k)$ and B is non-degenerate if and only if both maps are k -isomorphisms. Then B is symmetric if $B(m_1, m_2) = B(m_2, m_1)$ and symplectic if $B(m_1, m_1) = 0$ for all $m_1, m_2 \in M$. A subspace M_1 of M is totally isotropic if $B(m_1, m_2) = 0$ for all $m_1, m_2 \in M_1$. The space of bilinear forms on M can be naturally identified with $M^* \otimes M^*$.

Suppose that B is non-degenerate. Then its adjoint is the k -algebra anti-automorphism $f \rightarrow f^\sigma$, for $f \in E(M)$ defined by

$$B(m_1, fm_2) = B(f^\sigma m_1, m_2), \quad \text{for all } m_1, m_2 \in M.$$

So B is symmetric if and only if σ is an involution on $E(M)$, in which case $(E(M), \sigma)$ is an involutory k -algebra and we call (M, B) a symmetric or symplectic k -space.

An isometry is a k -linear map between symmetric or symplectic spaces which preserves the forms. An isometry is injective, but not necessarily surjective. Two symmetric or symplectic spaces are isomorphic if there is a surjective isometry between them. Two symmetric forms are isometric if the corresponding symmetric spaces are isomorphic. For example, B is isometric to $\lambda^2 B$, for all $\lambda \in k^\times$.

If B is symplectic, then it is alternating, meaning $B(m, m') = -B(m', m)$. Indeed alternating is the same as symplectic when $\text{char}(k) \neq 2$. But when $\text{char}(k) = 2$ alternating is the same as symmetric. So each symplectic form is symmetric, but there are symmetric forms which are not symplectic. For this reason we will not use the term alternating bilinear form.

Consider when k is algebraically closed and of characteristic 2. There is no standard name for a symmetric bilinear form which is not symplectic. If B is such a form, and B is non-degenerate, then M has an orthonormal basis with respect to B . So we call B a *diagonalizable* form. If instead B is symplectic, M has a symplectic basis with respect to B , and $\dim(M)$ is even. This discussion implies that, up to isometry, each k -vector space has one diagonalizable form, and an additional symplectic form, if its dimension is even.

Let $\text{GL}(M, B) = \{g \in \text{GL}(M) \mid B(gm_1, gm_2) = B(m_1, m_2), \forall m_1, m_2 \in M\}$ be the isometry group of a symmetric k -space (M, B) . Symplectic forms are more significant than diagonalizable forms. Using the methods of [12] it is not too hard to show that if k is a perfect field of characteristic 2 and $n = \dim(M)$ then

$$(1) \quad \text{GL}(M, B) \cong \begin{cases} \text{Sp}(n, k), & \text{if } B \text{ is symplectic, } n \text{ even,} \\ \text{Sp}(n-1, k), & \text{if } B \text{ is diagonalizable, } n \text{ odd,} \\ k^{n-1} : \text{Sp}(n-2, k), & \text{if } B \text{ is diagonalizable, } n \text{ even.} \end{cases}$$

Now suppose that M is a kG -module. Then B is G -invariant if $B(gm_1, gm_2) = B(m_1, m_2)$, for all $g \in G$. If in addition B is non-degenerate and symmetric or symplectic, we say that B is a G -form on M . We also say that M has symmetric or symplectic type. So (M, B) corresponds to a representation $G \rightarrow \text{GL}(M, B)$.

Consider the case where B is a symplectic G -form on M . Then there is a $\dim(M)$ space of quadratic forms on M which polarize to B . If one or more of these forms is G -invariant, we say that M has quadratic type.

Recall that $E(M)$ is a G -algebra, via ${}^g f(m) := gfg^{-1}(m)$, for $m \in M, f \in E(M)$ and $g \in G$. If B is a G -form, σ inverts the image of each $g \in G$ in $E(M)$. So

$$({}^g f)^\sigma = (gfg^{-1})^\sigma = gf^\sigma g^{-1} = {}^g(f^\sigma), \quad \text{for all } f \in E(M).$$

This implies that $(E(M), \sigma)$ is an involutory G -algebra.

As the Krull-Schmidt theorem fails to hold, we need to be able to distinguish between isomorphic submodules of M . A direct summand of M is a submodule M_1 of M such that $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$, for some submodule M_2 of M . We express this by writing $M = M_1 \dot{+} M_2$. We say that a kG -module L is a component of M , and write $L \mid M$, if L is isomorphic to a direct summand of M .

Now consider when B is a G -form on M . If M_1 is a submodule of M , then its orthogonal complement is the submodule $M_1^\perp := \{m \in M \mid B(M_1, m) = 0\}$. So B is non-degenerate on M_1 if and only if $M_1 \cap M_1^\perp = 0$. When this occurs $M = M_1 \dot{+} M_1^\perp$. Then we call M_1 a B -direct summand of M and write $M = M_1 \perp M_1^\perp$.

We say that a symmetric kG -module (L, B_L) is a component of (M, B) , and write $(L, B_L) \mid (M, B)$ if there is a kG -isometry $(L, B_L) \rightarrow (M, B)$. Note that then L is a component of M , as the image of an isometry is a direct summand of M .

The next result, which is well-known, implies that each indecomposable kG -module which affords a diagonalizable form belongs to the principal 2-block of G .

Lemma 2.4. *Let $\text{char}(k) = 2$ and suppose that M affords a G -form which is not symplectic. Then M has a trivial submodule and a trivial quotient module.*

Proof. We may assume that k is quadratically closed. Let B be a G -invariant symmetric bilinear form on M which is not symplectic. Set $q(m) = \sqrt{B(m, m)}$, for $m \in M$. Then $q : M \rightarrow k$ is a non-zero kG -homomorphism. So M has a trivial quotient module. Suppose that B is non-degenerate. Let η be the sum of the

vectors in any orthonormal basis for M . Then $q(m) = B(\eta, m)$, for all $m \in M$. As a consequence η spans a trivial submodule of M . \square

We will turn the somewhat anomalous nature of diagonalizable G -forms to our advantage in Section 5.

If $\text{char}(k) = 0$, each self-dual irreducible kG -module has symmetric or symplectic type. The type is detected by the Frobenius-Schur indicator of the associated ordinary irreducible character of G . This was generalised to $\text{char}(k) \neq 2$ by R. Gow. He noted that in that case each self-dual indecomposable kG -module has either symmetric or symplectic type. The problem of determining the type of a self-dual irreducible kG -module M was solved independently by W. Willems [30, 3.11] and J. G. Thompson [29]. The type is given by the Frobenius-Schur indicator of a real ordinary irreducible character of G which occurs with odd multiplicity in the principal indecomposable character of G corresponding to M .

When $\text{char}(k) \neq 2$, a symmetric form B can be conflated with the quadratic form $Q(m) := B(m, m)$, as $B(m_1, m_2) = \frac{1}{2}(Q(m_1 + m_2) - Q(m_1) - Q(m_2))$ is half the polarization of Q . However when $\text{char}(k) = 2$, the polarization of a quadratic form is symplectic, and there is a 1-parameter family of quadratic forms which polarize to a given symplectic form. In particular a G -invariant symplectic form need not be the polarization of a G -invariant quadratic form. It is an open problem to determine whether a self-dual irreducible kG -module has quadratic type in characteristic 2. The methods in this paper do not appear to have a bearing on this problem.

We note that when $\text{char}(k) = 2$, there are self-dual indecomposable kG -modules which are neither of symmetric or symplectic type. The unique non-trivial projective indecomposable $k(C_3 \rtimes C_4)$ -module is an example, as follows from [22, 1.3].

3. ADJOINTS

Throughout this section k is an algebraically closed field, M is a k -vector space, B is a non-degenerate symmetric or symplectic bilinear form on M and σ is the adjoint of B on $E(M)$.

3.1. Bilinear forms. For each $f \in E(M)$, define the bilinear form B_f on M as

$$(2) \quad B_f(m_1, m_2) := B(fm_1, m_2), \quad \text{for all } m_1, m_2 \in M.$$

Then $f \rightarrow B_f$ established a non-canonical k -isomorphism $E(M) \cong M^* \otimes_k M^*$ and

$$\begin{aligned} B_f \text{ is symmetric} &\iff f = f^\sigma, \\ B_f \text{ is symplectic} &\iff f = f_1 - f_1^\sigma, \text{ for some } f_1 \in E(M), \\ B_f \text{ is non-degenerate} &\iff f \in \text{GL}(M). \end{aligned}$$

The adjoint of a non-degenerate symmetric or symplectic bilinear form is an involution on $E(M)$. Conversely, each involution on $E(M)$ corresponds to a 1-parameter family of such forms on M , as we now show:

Lemma 3.1. (i) Let τ be an involution on $E(M)$. Then up to a non-zero scalar, M has a unique non-degenerate symmetric or symplectic form whose adjoint is τ .
(ii) Let $M = M_1 \oplus M_2$ and let τ be an involution on $E(M_1) \times E(M_2)$ which interchanges $E(M_1)$ with $E(M_2)$. Then τ has a unique extension to an involution on $E(M)$. The corresponding forms on M are symplectic and totally isotropic on M_1 and M_2 .

Proof. The hypothesis (ii) implies that $\dim(M_1) = \dim(M_2) = \frac{1}{2} \dim(M)$. So in case (ii) we can and do choose B so that $B(m_i, m'_i) = 0$, for all $m_i, m'_i \in M_i$, for $i = 1, 2$. In particular B is symplectic.

Assume the hypothesis (i). Then $\tau\sigma$ is a k -algebra automorphism of $E(M)$. So by the Skolem-Noether theorem there is $g \in \text{GL}(M)$ such that $f^{\tau\sigma} = gfg^{-1}$, for all $f \in E(M)$. So $f^\tau = g^{-\sigma} f^\sigma g^\sigma$. Then B_g is a non-degenerate symmetric form whose adjoint is τ . Moreover g , and thus B_g , is determined up to a scalar.

Assume the hypothesis (ii). Let e_1, e_2 be idempotents in $E(M)$ such that $1_M = e_1 + e_2$, $e_i M = M_i$ and $\ker(e_i) = M_{3-i}$. Then $E(M_1) \times E(M_2)$ embeds in $E(M)$ as $e_1 E(M) e_1 + e_2 E(M) e_2$. Now $\tau\sigma$ maps each $E(M_i)$ onto itself and hence restricts to an automorphism on the semi-simple k -algebra $E(M_1) \times E(M_2)$. Again by the Skolem-Noether theorem there exists $g_1 \in \text{GL}(M_1)$, and $g_2 \in \text{GL}(M_2)$ such that $(f_1 + f_2)^{\tau\sigma} = (g_1 f_1 g_1^{-1} + g_2 f_2 g_2^{-1})$, for all $f_1 \in E(M_1)$ and $f_2 \in E(M_2)$. Moreover each of g_1 and g_2 is determined up to a scalar. Applying σ to both sides, we get $(f_1 + f_2)^\tau = g_2^{-\sigma} f_2^\sigma g_2^\sigma + g_1^{-\sigma} f_1^\sigma g_1^\sigma$. Then

$$f_1 + f_2 = ((f_1 + f_2)^\tau)^\tau = g_2^{-\sigma} g_1 f_1 g_1^{-1} g_2^\sigma + g_1^{-\sigma} g_2 f_2 g_2^{-1} g_1^\sigma.$$

This holds for all $f_1 \in E(M_1)$. So there is $\lambda \in k^\times$ such that $g_2^\sigma = \lambda g_1$. Thus $g_1^\sigma = \lambda^{-1} g_2$. Now replace g_2 by $\lambda^{-1} g_2$. Then $g_2^\sigma = g_1$ and

$$(f_1 + f_2)^\tau = g_1^{-1} f_2^\sigma g_1 + g_2^{-1} f_1^\sigma g_2, \quad \text{for all } f_1 \in E(M_1), f_2 \in E(M_2).$$

Note that $g_1 + g_2 \in \text{GL}(M)$. Then $B_{g_1+g_2}$ is a symplectic form on M whose adjoint extends τ to $E(M)$. Moreover no other involution on $E(M)$ extends τ . \square

3.2. Endomorphisms. Assume now that M is a kG -module and B is G -invariant. We use $E_G(M)$ to denote the ring of kG -endomorphisms of M . So $E_G(M)$ is a unital subalgebra of $E(M)$. The G -invariant bilinear forms on M are B_f for $f \in E_G(M)$, where B_f is symmetric if and only if $f = f^\sigma$. The following is very useful:

Lemma 3.2. *Suppose that M is a non-degenerate indecomposable component of $(L_1, B_1) \perp \cdots \perp (L_n, B_n)$, where each (L_i, B_i) is a symmetric kG -module. Then M is a non-degenerate component of some (L_i, B_i) .*

Proof. By hypothesis there is an isometry $\alpha : (M, B_f) \rightarrow (L_1, B_1) \perp \cdots \perp (L_n, B_n)$ for some σ -invariant unit $f \in E_G(M)$. Now there are kG -maps $\alpha_i : M \rightarrow L_i$ with

$$\alpha(m) = \alpha_1(m) + \cdots + \alpha_n(m), \quad \text{for all } m \in M.$$

Each $(m_1, m_2) \rightarrow B_i(\alpha_i m_1, \alpha_i m_2)$ is a G -invariant symmetric bilinear form on M . So there are σ -invariant $\gamma_i \in E_G(M)$ such that

$$B_i(\alpha_i m_1, \alpha_i m_2) = B_{\gamma_i}(m_1, m_2).$$

Now $\gamma_1 + \cdots + \gamma_n = f$, as α is an isometry. But $E_G(M)$ is a local ring, as M is indecomposable. So γ_j is a unit, for some j . Then B_{γ_j} is non-degenerate on M and $\alpha_j : (M, B_{\gamma_j}) \rightarrow (L_j, B_j)$ is a kG -isometry. \square

Our next result should be well-known:

Lemma 3.3. *Suppose that $\theta \in E(M)$. Then $(\theta M)^\perp = \ker(\theta^\sigma)$. So B is non-degenerate on θM if and only if θ^σ restricts to an isomorphism $\theta M \rightarrow \theta^\sigma M$.*

Proof. The first statement holds because $m \in (\theta M)^\perp \iff B(\theta^\sigma m, m') = 0$, for all $m' \in M$. Then $\theta M \cap (\theta M)^\perp = 0$ if and only if θ^σ is injective on θM . But $\dim(\theta M) = \dim(\theta^\sigma M)$. So $\theta^\sigma : \theta M \rightarrow \theta^\sigma M$ is injective if and only if it is surjective. \square

Next recall that a pairing between two k -vector spaces L and M is a bilinear map $L \times M \rightarrow k$. The pairing is perfect if $\ell \mapsto B(\ell, -)$ for $\ell \in L$, induces a k -isomorphism $L \rightarrow M^*$. The canonical example is the perfect pairing $P : M^* \times M \rightarrow k$ given by $P(f, m) = f(m)$, for $f \in M^*$ and $m \in M$. Then P extends to a non-degenerate symplectic bilinear form on $M^* \oplus M$, also denoted by P , which is zero on restriction to each of M^* and M . We call $(M^* \oplus M, P)$ the paired module on M .

Recall that M^* is a kG -module via $(gf)(m) := f(g^{-1}m)$, for $g \in G, f \in M^*$ and $m \in M$. A perfect G -pairing between two kG -modules L and M is a perfect pairing P for which $P(g\ell, gm) = P(\ell, m)$, for all $\ell \in L, m \in M$ and $g \in G$. Equivalently $\ell \mapsto B(\ell, -)$ for $\ell \in L$, is a kG -module isomorphism $L \cong M^*$. For example $P : M^* \times M \rightarrow k$ is a perfect G -pairing.

We now associate to each G -endomorphism of M a perfect G -pairing on a pair of submodules of M . This construction is essential to our characterisation of projectivity of forms in Lemma 4.3.

Lemma 3.4. *Given $\theta \in E_G(M)$ define $\hat{B}_\theta : \theta M \times \theta^\sigma M \rightarrow k$ via*

$$\hat{B}_\theta(\theta m_1, \theta^\sigma m_2) = B(\theta m_1, m_2), \quad \text{for all } m_1, m_2 \in M.$$

Then \hat{B}_θ is a perfect G -pairing, and thus $\theta^\sigma M \cong (\theta M)^$ as kG -modules.*

Conversely if P is a perfect G -pairing between submodules L_1 and L_2 of M then there is a unique $\psi \in E_G(M)$ such that $L_1 = \psi M, L_2 = \psi^\sigma M$ and $P = \hat{B}_\psi$.

Proof. We note that \hat{B}_θ is well-defined as $B(\theta m_1, m_2) = B(m_1, \theta^\sigma m_2)$. Lemma 3.3 implies that \hat{B}_θ is a perfect G -pairing between θM and $\theta^\sigma M$.

Let $P : L_1 \times L_2 \rightarrow k$ be a perfect G -pairing. Then for each $m \in M$ there is $\psi m \in L_1$ such that $B(m, \ell_2) = P(\psi m, \ell_2)$ for all $\ell_2 \in L_2$. Check that $\psi \in E_G(M)$ and $\psi M = L_1$. Likewise there is $\psi' \in E_G(M)$ such that $B(\ell_1, m) = P(\ell_1, \psi' m)$ for all $\ell_1 \in L_1$. Now for all $m_1, m_2 \in M$

$$B(m_1, \psi' m_2) = P(\psi m_1, \psi' m_2) = B(\psi m_1, m_2).$$

It follows that $\psi' = \psi^\sigma$ and $P = \hat{B}_\psi$. \square

Note that $\theta^\sigma = \theta$ if and only if $\theta M = \theta^\sigma M$ and \hat{B}_θ is symmetric.

3.3. Idempotents. We continue with our assumption that M is a kG -module and B is G -invariant. Now if $M = M_1 \dot{+} \dots \dot{+} M_t$ as kG -modules, then the projections onto the M_i are pairwise orthogonal idempotents in $E_G(M)$ which sum to 1_M . In the presence of the G -form B , we need to consider self-adjoint idempotents:

Lemma 3.5 (Orthogonal Projection). *Let L be a kG -submodule of M . Then B is non-degenerate on L if and only if $L = eM$, for some σ -invariant idempotent $e \in E(M)$. If e exists it is unique, G -invariant and $\ker(e) = L^\perp$.*

Proof. Suppose that e exists. Then $e^\sigma M = e^\sigma e M = e M$. So B is non-degenerate on L , by Lemma 3.3. Moreover $\ker(e) = (e^\sigma M)^\perp = L^\perp$. This ensures that e is unique, and this forces $e \in E_G(M)$.

Conversely, suppose that $L \cap L^\perp = 0$. Let e be projection onto L with kernel L^\perp . Then $\ker(e^\sigma) = (eM)^\perp = L^\perp$ and $e^\sigma M = \ker(e)^\perp = L$. So e^σ is projection onto L with kernel L^\perp . We deduce that $e^\sigma = e$. \square

Each G -invariant form on a direct summand of M extends to M , as we show:

Lemma 3.6. *Let $e \in E_G(M)$ be idempotent. Then each G -invariant form on eM is the restriction of a G -invariant form B_θ on M , for some $\theta \in e^\sigma E_G(M)e$.*

Proof. Using Lemma 3.4, $e^\sigma E(M)e = \text{Hom}(eM, e^\sigma M) \cong (eM)^* \otimes (eM)^*$. Also B_ϕ and $B_{e^\sigma \phi e}$ have the same restrictions to eM , for all $\phi \in E_G(M)$.

Let \hat{B} be a G -invariant bilinear form on eM . Then $\hat{B}(e_-, e_-)$ defines a G -invariant bilinear form on M . So there exists $\theta \in e^\sigma E_G(M)e$ with $\hat{B}(em_1, em_2) = B_\theta(m_1, m_2)$, for all $m_1, m_2 \in M$. \square

Our next result is required in Proposition 5.16. The proof uses ideas from [8]:

Lemma 3.7. *Let A be a semi-simple subalgebra of $E(M)$ and let $f \in A \cap A^\sigma$ be an idempotent such that fM is a submodule of M and B is non-degenerate on fM . Let e be orthogonal projection onto fM . Then $e \in E_G(M) \cap A$.*

Proof. Lemma 3.3 implies that $f^\sigma fM = f^\sigma M$. So by the Artin-Wedderburn Theorem and the Jacobson Density Lemma $f^\sigma = f^\sigma f a$ for some $a \in A$. Set $e = f a f^\sigma$. Then $e \in A$, by hypothesis on f and

$$e^\sigma e = (f a^\sigma f^\sigma)(f a f^\sigma) = f a^\sigma (f^\sigma f a) f^\sigma = f a^\sigma (f^\sigma)^2 = e^\sigma.$$

So $e = (e^\sigma e)^\sigma = (e^\sigma e) = e^\sigma$. Then $e^2 = e^\sigma e = e$. Moreover $e \in A$.

Now $eM = fM$, as $e = f e$ and $f = f f = f(a^\sigma f^\sigma f) = (f a^\sigma f^\sigma) f = e f$. As e is orthogonal projection onto fM , Lemma 3.5 implies that $e \in E_G(M)$. \square

Finally we prove [9, Proposition] using methods which will be developed later:

Lemma 3.8. *Suppose that $M = M_1 \dot{+} \dots \dot{+} M_t$ where each M_i is an indecomposable kG -module. Then for each i*

- (i) B is non-degenerate on M_i or
- (ii) B is non-degenerate on $M_i \dot{+} M_j$ for some $j \neq i$ with $M_j \cong M_i^*$.

Proof. Write $1_M = e_1 + \dots + e_t$ where e_1, \dots, e_t are pairwise orthogonal primitive idempotents in $E_G(M)$ with $e_j M = M_j$. Let ϵ be the point of $E_G(M)$ containing e_i . We use $\bar{}$ for images in $E(P_\epsilon)$.

Suppose first that $M_i \cong M_i^*$. Then $\epsilon^\sigma = \epsilon$, using Lemma 3.4. So $(E(P_\epsilon), \sigma)$ is an involutory k -algebra. Lemma 3.1 implies that P_ϵ affords a symmetric form B_ϵ with adjoint σ . For all j with $e_j \in \epsilon$, choose $s_j \in \bar{e_j} P_\epsilon$ with $s_j \neq 0$. Then the s_j form a basis of P_ϵ .

Say $B_\epsilon(s_i, s_i) \neq 0$. Then B_ϵ is non-degenerate on ks_i . Let $x_i \in \bar{e_i} E(P_\epsilon)$ be orthogonal projection onto ks_i , as given by Lemma 3.5. By Lemma 2.3, there is a σ -invariant idempotent $\bar{f_i} \in e_i E_G(M) e_i^\sigma$ such that $\bar{f_i} = x_i$. Then $f_i M \subseteq M_i$ and M_i is indecomposable. So $f_i M = M_i$. It then follows from Lemma 3.5 that B is non-degenerate on M_i .

Now say $B_\epsilon(s_i, s_i) = 0$. As B_ϵ is non-degenerate, we may choose $j \neq i$ such that $B_\epsilon(s_i, s_j) \neq 0$. So B_ϵ is non-degenerate on $ks_i + ks_j$. Let $x_{ij} \in \bar{e_i + e_j} E(P_\epsilon)$ be orthogonal projection onto $ks_i + ks_j$, as given by Lemma 3.5. By Lemma 2.3, there is a σ -invariant idempotent $\bar{f_{ij}} \in (e_i + e_j) E_G(M)$ such that $\bar{f_{ij}} = x_{ij}$. Then

$f_{ij}M = M_i + M_j$, as $f_{ij}M \subseteq M_i + M_j$ but f_{ij} is not primitive. So B is non-degenerate on $M_i + M_j$, by Lemma 3.5.

Finally, suppose that $M_i \not\cong M_i^*$. Then $\epsilon^\sigma \neq \epsilon$ and $(E(P_\epsilon) \times E(P_{\epsilon^\sigma}), \sigma)$ is an involutory k -algebra satisfying the hypothesis of Lemma 3.1(ii). Let $B_{\epsilon, \epsilon^\sigma}$ be the corresponding symplectic form on $P_\epsilon \oplus P_{\epsilon^\sigma}$. We proceed as above; there exists j such that $e_j \in \epsilon^\sigma$ and $B_{\epsilon, \epsilon^\sigma}$ is non-degenerate on $\pi_{\epsilon, \epsilon^\sigma}(e_i + e_j)(P_\epsilon \oplus P_{\epsilon^\sigma})$. Then there is a σ -invariant idempotent $f_{ij} \in (e_i + e_j)E_G(M)$ such that $f_{ij}M = M_i + M_j$. So B is non-degenerate on $M_i + M_j$, and $M_j \cong M_i^*$. \square

3.4. Projective Representations. For the rest of Section 3, k has characteristic 2 and M is a k -vector space. The map $f \rightarrow gfg^{-1}$, for $g \in \text{GL}(M)$, is a k -algebra automorphism of $E(M)$. Using this, we can identify $\text{PGL}(M) = \text{GL}(M)/k^\times 1_M$ with $\text{Aut}(E(M))$. As σ is a k -algebra anti-automorphism of $E(M)$, it acts on $\text{PGL}(M)$. We set $\text{PGL}(M, \sigma)$ as the centralizer of σ in $\text{PGL}(M)$.

Lemma 3.9. *Each projective representation $\theta : G \rightarrow \text{PGL}(M, \sigma)$ has a unique lift to a group representation $G \rightarrow \text{GL}(M, B)$.*

Proof. Let $\rho : \text{GL}(M) \rightarrow \text{PGL}(M)$ be the projection, with kernel $k^\times 1_M$. If $g \in \text{GL}(M, B)$, then $\rho(g) \in \text{PGL}(M, \sigma)$. Suppose that $\lambda 1_M \in \text{GL}(M, B)$, with $\lambda \in k^\times$. Then $(\lambda 1_M)^\sigma = \lambda^{-1} 1_M$. But $(\lambda 1_M)^\sigma = \lambda 1_M$, as σ is k -linear. So $\lambda = \lambda^{-1}$. As $\text{char}(k) = 2$, it follows that $\lambda = 1_k$. This shows that ρ restricts to an injective map $\hat{\rho} : \text{GL}(M, B) \rightarrow \text{PGL}(M, \sigma)$. We claim that $\hat{\rho}$ is surjective. To see this, let $a \in \text{PGL}(M, \sigma)$. Then $a = \rho(g)$, for some $g \in \text{GL}(M)$. Now

$$g^{-\sigma} f g^\sigma = ({}^a f)^\sigma = {}^a f = g f g^{-1}, \quad \text{for all } f \in E(M).$$

So $g^\sigma = \lambda g^{-1}$ for some $\lambda \in k^\times$, by the Skolem-Noether theorem. Then $\sqrt{\lambda^{-1}}g \in \text{GL}(M, B)$ and $\hat{\rho}(\sqrt{\lambda^{-1}}g) = a$. Our claim follows from this.

For uniqueness, suppose that X and Y are representations $G \rightarrow \text{GL}(M, B)$ lifting θ . Then there is a function $\gamma : G \rightarrow k^\times$ such that $Y(g) = \gamma(g)X(g)$, for all $g \in G$. Applying σ to both sides we get $Y(g^{-1}) = \gamma(g)X(g^{-1})$. But $Y(g^{-1}) = Y(g)^{-1} = \gamma(g)^{-1}X(g^{-1})$. Comparing, we see that $\gamma(g)^{-1} = \gamma(g)$. So $\gamma(g) = 1_k$ and $X(g) = Y(g)$, for all $g \in G$. \square

We recall some results from [22, Appendix A]. Suppose that e_1 and e_2 are orthogonal idempotents in $E(M)$ such that $1_M = e_1 + e_2$ and $e_1^\sigma = e_2$. Set $M_i = e_i M$. So $M = M_1 \oplus M_2$ and B is zero on each of M_1 and M_2 and is symplectic on M . The stabilizer of $\{M_1, M_2\}$ in $\text{GL}(M)$ is the group $\text{GL}(M_1, M_2) = \text{GL}(M_1) \times \text{GL}(M_2) : \langle s \rangle$. Here we can and do assume that s is an involution which interchanges M_1 and M_2 .

Let $\text{PGL}(M_1, M_2) \cong \text{PGL}(M_1) \wr S_2$ be the group of automorphisms of the semi-simple k -algebra $E(M_1) \times E(M_2)$. Now $\text{GL}(M_1, M_2)$ acts by conjugation on $E(M_1) \times E(M_2)$ and the resulting map $\phi : \text{GL}(M_1, M_2) \rightarrow \text{PGL}(M_1, M_2)$ is surjective. Moreover $\ker(\phi) = \{ae_1 + be_2 \mid a, b \in k^\times\} \cong k^\times \times k^\times$.

Set $\text{Sp}(M_1, M_2) = \text{GL}(M, B) \cap \text{GL}(M_1, M_2)$. If τ is transposition then

$$\text{Sp}(M_1, M_2) = \{(g, sg^{-1}s) \in \text{GL}(M_1) \times \text{GL}(M_2) \mid sgs = g^{-\tau}\} : \langle s \rangle$$

Now let $\text{PGL}(M_1, M_2, \sigma)$ be the centralizer of σ in $\text{PGL}(M_1, M_2)$. Then ϕ restricts to a surjective map $\theta : \text{Sp}(M_1, M_2) \rightarrow \text{PGL}(M_1, M_2, \sigma)$. The kernel of θ is $K = \{(a, a^{-1}) \mid a \in k^\times\}$. So $K \cong k^\times$ and s inverts each element of K .

Lemma 3.10. *Each projective representation $\rho : G \rightarrow \mathrm{PGL}(M_1, M_2, \sigma)$ is realised by a group representation $\chi : H \rightarrow \mathrm{Sp}(M_1, M_2)$ which arises from a commutative diagram of finite groups with exact rows*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & O & \xrightarrow{\mathrm{inc}} & H & \xrightarrow{\nu} & G & \longrightarrow & 1 \\ & & \downarrow \eta & & \downarrow \chi & & \downarrow \rho & & \\ 1 & \longrightarrow & K & \xrightarrow{\mathrm{inc}} & \mathrm{Sp}(M_1, M_2) & \xrightarrow{\theta} & \mathrm{PGL}(M_1, M_2, \sigma) & \longrightarrow & 1 \end{array}$$

Here O is a cyclic group of odd order, η is injective, $[H : C_H(O)] \leq 2$ and all elements of $H \setminus C_H(O)$ invert O .

Proof. The pull-back diagram associated with ρ and θ is

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\mathrm{inc}} & \hat{G} & \xrightarrow{\nu} & G & \longrightarrow & 1 \\ & & \downarrow = & & \downarrow \chi & & \downarrow \rho & & \\ 1 & \longrightarrow & K & \xrightarrow{\mathrm{inc}} & \mathrm{Sp}(M_1, M_2) & \xrightarrow{\theta} & \mathrm{PGL}(M_1, M_2, \sigma) & \longrightarrow & 1 \end{array}$$

Every element of G centralizes or inverts K . In this way $K \cong k^\times$ is a (possibly non-trivial) $\mathbb{Z}G$ -module. Set $\gamma(\lambda) = \lambda^{|G|}$, for all $\lambda \in K$. As k is algebraically closed, γ is a surjective endomorphism of K . We have a short exact sequence of abelian groups

$$1 \longrightarrow O \xrightarrow{\eta} K \xrightarrow{\gamma} K \longrightarrow 1.$$

Here O is the set of roots of $x^{|G|} - 1$ in k . So O is a finite group. This induces a long exact sequence in cohomology, including

$$\dots \longrightarrow H^2(G, O) \xrightarrow{\eta_*} H^2(G, K) \xrightarrow{\gamma_*} H^2(G, K) \longrightarrow \dots$$

Now γ_* is the zero map, as multiplication by $|G|$ annihilates $H^2(G, K)$. Let $d \in H^2(G, K)$ be the factor set associated with

$$1 \longrightarrow K \xrightarrow{\mathrm{inc}} \hat{G} \xrightarrow{\nu} G \longrightarrow 1.$$

Then there exists $c \in H^2(G, O)$ mapping onto d . This gives us the commutative diagram in the statement of the Lemma. \square

We mention that Theorem A.5 in [22] wrongly claims (in the notation used here) that H is a central extension of G . Now [22, 7.2] relies on Theorem A.5, but does not require $O \leq Z(H)$. So 7.2 is still correct.

4. INDUCTION AND BILINEAR FORMS

Throughout this section M is a kG -module, B is a symmetric G -form on M and σ is the adjoint of B on $E(M)$. In many results we could take B to be symplectic. We will require $\mathrm{char}(k) = 2$ in part 4.3.

4.1. Induction and Mackey's Formula. For H a subgroup of G the trace map $\text{tr}_H^G : E_H(M) \rightarrow E_G(M)$ is the k -linear map $\text{tr}_H^G(f) := \sum {}^g f$, for $f \in E_H(M)$, where g ranges over a left transversal G/H to H in G . Its image is an ideal of $E_G(M)$, denoted $E_H^G(M)$. The σ -invariant elements in $E_H(M)$ form a subspace, but not an ideal, of $E_H(M)$.

If L is a kH -module, the induced module $\text{Ind}_H^G L$ is a direct sum of the k -vector spaces ${}^g L$, as g ranges over G/H . Here ${}^g L = g \otimes L$ is the $k^g H$ -module such that $ghg^{-1}(g \otimes \ell) = g \otimes h\ell$, for all $h \in H$ and $\ell \in L$. Recall that M is said to be H -projective if $M \mid \text{Ind}_H^G L$, for some kH -module L .

Let B_L be a symmetric H -form on L . The induced symmetric kG -module $\text{Ind}_H^G(L, B_L)$ is $\text{Ind}_H^G L$ with the induced G -form B_L^G , where

$$B_L^G(g_1 \otimes \ell_1, g_2 \otimes \ell_2) = \begin{cases} B_L(g_2^{-1}g_1\ell_1, \ell_2), & \text{if } g_1H = g_2H. \\ 0, & \text{if } g_1H \neq g_2H. \end{cases}$$

Let ${}^g B_L$ denote the restriction of B_L^G to ${}^g L$. Then $\text{Ind}_H^G(L, B_L)$ is the orthogonal direct sum of the symmetric k -spaces $({}^g L, {}^g B_L)$. It is clear that B_L^G is symplectic if and only if B_L is symplectic.

There is a version of Mackey's formula [24, 3.1.9] for symmetric modules. However its usefulness is limited by the absence of the Krull-Schmidt theorem:

Lemma 4.1. *Given $K \leq G$, there is an isomorphism of symmetric kK -modules*

$$\text{Res}_K^G \text{Ind}_H^G(L, B_L) \cong \bigoplus_{g \in K \backslash G/H} \text{Ind}_{K \cap {}^g H}^K \text{Res}_{K \cap {}^g H}^{{}^g H}({}^g L, {}^g B_L).$$

Proof. Let $g \in G$. The assignment $i \rightarrow ig$ maps $K/K \cap {}^g H$ to a set of representatives for the left cosets of H in KgH and $\sum_{i \in K/K \cap {}^g H} {}^i g L \cong \text{Ind}_{K/K \cap {}^g H}^K {}^i g L$. This induces an isomorphism $\sum_{i \in K/K \cap {}^g H} ({}^i g L, {}^i g B_L) \cong \text{Ind}_{K/K \cap {}^g H}^K ({}^g L, {}^g B_L)$ of symmetric kK -modules. \square

4.2. Higman's Criterion. Higman's Criterion [24, 4.2.2] is the assertion that the following are equivalent:

- (1) M is H -projective.
- (2) $\text{tr}_H^G(\alpha)$ is a unit in $E_G(M)$, for some $\alpha \in E_H(M)$.
- (3) $M \mid \text{Ind}_H^G \text{Res}_H^G M$.

We will generalise the equivalence (1) \iff (2) to symmetric kG -modules. However, the analogue of (3) is a strictly stronger statement.

Recall from Section 3.2 that $\{B_\theta \mid \theta \in E_G(M)^\times, \theta^\sigma = \theta\}$ give all non-degenerate G -invariant symmetric bilinear forms on M . Here are some important definitions:

Definition 4.2. *Let θ be a σ -invariant unit in $E_G(M)$. Then we say that*

- θ is (H, σ) -projective if $\theta = \text{tr}_H^G(\alpha)$ for some $\alpha \in E_H(M)$ with $\alpha^\sigma = \alpha$.
- B_θ is H -projective if (M, B_θ) is a component of $\text{Ind}_H^G(L, B_L)$ for some symmetric kH -module (L, B_L) .
- M is symmetrically H -projective if it affords a non-degenerate G -invariant symmetric bilinear form which is H -projective.

We note that if B_θ is H -projective, then it is projective relative to every subgroup of G containing a G -conjugate of H .

Lemma 4.3. *Let θ be a σ -invariant unit in $E_G(M)$. Then θ is (H, σ) -projective if and only if B_θ is H -projective. More precisely, if $\theta = \text{tr}_H^G(\alpha)$ for some σ -invariant $\alpha \in E_H(M)$, then there is a kG -isometry $(M, B_\theta) \rightarrow \text{Ind}_H^G(\alpha M, \hat{B}_\alpha)$.*

Proof. Here \hat{B}_α is the H -form on αM defined in Lemma 3.4.

Suppose first that $\theta = \text{tr}_H^G(\alpha)$ for some σ -invariant $\alpha \in E_H(M)$. Then $1 \otimes \alpha$ can be regarded as a kH -homomorphism $M \rightarrow \text{Ind}_H^G(\alpha M)$. So $\phi = \text{tr}_H^G(1 \otimes \alpha)$ is a kG -homomorphism $M \rightarrow \text{Ind}_H^G(\alpha M)$. Now

$$\begin{aligned} \hat{B}_\alpha(\phi m_1, \phi m_2) &= \sum_{g \in G/H} \hat{B}_\alpha(\alpha g m_1, \alpha g m_2) = \sum_{g \in G/H} B(\alpha g m_1, g m_2) \\ &= B(\text{tr}_H^G(\alpha) m_1, m_2) = B_\theta(m_1, m_2). \end{aligned}$$

So ϕ defines an isometry $(M, B_\theta) \rightarrow \text{Ind}_H^G(\alpha M, \hat{B}_\alpha)$. Thus B_θ is H -projective.

Now suppose that B_θ is H -projective. Then there is a kG -isometry $\phi : (M, B_\theta) \rightarrow \text{Ind}_H^G(L, B_1)$ for some symmetric kH -module (L, B_1) . Let $e \in E_H(\text{Ind}_H^G L)$ be the orthogonal projection onto $1 \otimes L$. Now $B_1(e\phi_-, e\phi_-)$ is a symmetric H -form on M . So there exists $\alpha \in E_H(M)$ such that $\alpha^\sigma = \alpha$ and

$$B_1(e\phi m_1, e\phi m_2) = B(\alpha m_1, m_2), \quad \text{for all } m_1, m_2 \in M.$$

As ϕ is an isometry, we have

$$\begin{aligned} B_\theta(m_1, m_2) &= B_1^G(\phi m_1, \phi m_2) = \sum_{g \in G/H} B_1(e\phi g m_1, e\phi g m_2) \\ &= \sum_{g \in G/H} B(\alpha g m_1, g m_2) = B(\text{tr}_H^G(\alpha) m_1, m_2). \end{aligned}$$

So $\theta = \text{tr}_H^G(\alpha)$ is (H, σ) -projective. \square

We now prove an analogue of Higman's Criterion for symmetric modules:

Proposition 4.4. *The following are equivalent:*

- (1) M is symmetrically H -projective.
- (2) $\text{tr}_H^G(\alpha)$ is a unit in $E_G(M)$, for some σ -invariant $\alpha \in E_H(M)$.
- (2)' B_θ is H -projective for some σ -invariant unit $\theta \in E_G(M)$.

Proof. (1) and (2)' are equivalent from the definitions. Lemma 4.3 shows that (2) and (2)' are equivalent. \square

Be aware that if (M, B_θ) is a component of $\text{Ind}_H^G \text{Res}_H^G(M, B_\theta)$ then it does not follow that it is a component of $\text{Ind}_K^G \text{Res}_K^G(M, B_\theta)$ when $H \leq K \leq G$. So we merely state the following analogue of condition (3) in Higman's criterion:

Lemma 4.5. *Let θ be a σ -invariant unit in $E_G(M)$. Then*

$$(M, B_\theta) \mid \text{Ind}_H^G \text{Res}_H^G(M, B_\theta)$$

if and only if $\text{tr}_H^G(\alpha \theta \alpha^\sigma) = \theta$, for some $\alpha \in E_H(M)$.

4.3. Green and Symmetric Vertices. Now assume that M is indecomposable and $\text{char}(k) = 2$. We prove the first statement of Theorem 1.2. The main technical problem is that trace maps do not behave well with respect to adjoints. So Mackey's formula for the product of a pair of traces is not useful in our context. Instead we cancel pairs of terms in *triple* products of traces.

Lemma 4.6. *Each symmetric vertex of M contains a Green vertex of M with index at most 2.*

Proof. Let T be a symmetric vertex of M and let V be a Green vertex of M which is contained in T . There is nothing to prove if $V = T$. So assume that $V \neq T$.

By Higman's criterion $1_M = \text{tr}_V^G(\alpha)$, for some $\alpha \in E_V(M)$. Now let θ be a σ -invariant unit in $E_G(M)$ which is (T, σ) -projective. So $\theta = \text{tr}_T^G(\beta)$ for some σ -invariant $\beta \in E_T(M)$. Consider the triple product

$$\theta = \text{tr}_V^G(\alpha) \text{tr}_T^G(\beta) \text{tr}_V^G(\alpha^\sigma) = \sum_{a, b \in G/V, c \in G/T} ({}^a\alpha)({}^c\beta)({}^b\alpha^\sigma).$$

Each G -orbit in $G/V \times G/T \times G/V$ contains a 3-tuple of the form (aV, T, bV) . We say that the orbit is:

- diagonal if $aV = bV$,
- symmetric if $aV \neq bV$ but the orbit contains (bV, T, aV) ,
- antisymmetric if the orbit does not contain (bV, T, aV) .

We denote the collections of such orbits by $\mathcal{O}_d, \mathcal{O}_s$ and \mathcal{O}_a respectively.

The stabilizer of (aV, T, bV) is ${}^aV \cap {}^bV \cap T$. So the orbit sum is

$$\text{tr}(a, b) := \text{tr}_{aV \cap bV \cap T}^G({}^a\alpha \beta({}^b\alpha^\sigma)) \in E_G(M).$$

Now $\text{tr}(a, b)^\sigma = \text{tr}(b, a)$. So θ is a sum, in $E_G(M)$, of σ -invariant terms

$$\theta = \sum_{\mathcal{O}_d} \text{tr}(a, a) + \sum_{\mathcal{O}_s} \text{tr}(a, b) + \sum_{\mathcal{O}_a} (\text{tr}(a, b) + \text{tr}(b, a)).$$

Write $\text{tr}(a, b) = \lambda 1_M + j$, with $\lambda \in k^\times$ and $j \in J(E_G(M))$. Then for each pair of anti-symmetric orbits $\text{tr}(a, b) + \text{tr}(b, a) = j + j^\sigma$ belongs to $J(E_G(M))$. Suppose that $\text{tr}(a, a)$ is a unit in $E_G(M)$, for some diagonal orbit. Then $B_{\text{tr}(a, a)}$ is a $({}^aV \cap T)$ -projective symmetric G -form on M . This is impossible, as ${}^aV \cap T \not\leq T$.

Now θ is a unit in the local ring $E_G(M)$. So we can choose a triple (aV, T, bV) in a symmetric orbit such that $\text{tr}(a, b)$ is a unit. We then replace V by a conjugate so that $a = 1$, to simplify the notation. Then $\text{tr}(1, b)$ is a unit and (V, T, bV) is in a symmetric G -orbit.

As (bV, T, V) is G -conjugate to (V, T, bV) there is $t \in T$ with $tV = bV$ and $tbV = V$. So $t \in N_T(V \cap {}^bV)$ and $t^2 \in V \cap {}^bV$. Then $VbV = Vb^{-1}V$ is a self-dual double coset and $[(V \cap {}^bV)\langle t \rangle : V \cap {}^bV] = 2$.

Set $\gamma := \alpha \beta({}^b\alpha^\sigma)$. Then $\gamma^\sigma = {}^b\alpha \beta \alpha^\sigma = \gamma^t$. So $\gamma + {}^t\gamma$ is fixed by both σ and $(V \cap {}^bV)\langle t \rangle \cap T$. Moreover $\text{tr}(1, b) = \text{tr}_{(V \cap {}^bV)\langle t \rangle \cap T}^G(\gamma + {}^t\gamma)$. As T is a symplectic vertex of M , this forces $(V \cap {}^bV)\langle t \rangle = T$. But $V \neq T$ and $t^2 \in V$. We deduce that $V = {}^bV$, $T = V\langle t \rangle$ and $[T : V] = 2$. \square

We will use our second result in Proposition 5.13.

Lemma 4.7. *Suppose that M is symmetrically H -projective. Then M is a non-degenerate component of some $\text{Ind}_H^G(L, B_L)$ where L is indecomposable.*

Proof. There is a kG -isometry $\phi : (M, B_\theta) \rightarrow \text{Ind}_H^G(L, B_L)$, where (L, B_L) is a symmetric kH -module and θ is a σ -invariant unit in $E_G(M)$. Choose this ϕ with $\dim(L)$ minimal. So (L, B_L) is orthogonally indecomposable, by Lemma 3.2.

We claim that L is indecomposable. For otherwise $L = L_1 \dot{+} L_2$, with $L_1^* \cong L_2$, according to Lemma 3.8. Let $e \in E_H(\text{Ind}_H^G L)$ be orthogonal projection onto $1 \otimes L$ and let $e_i = ee_i e$ be projection onto $1 \otimes L_i$ with kernel $1 \otimes L_{3-i}$, for $i = 1, 2$. Now there are $\alpha_{ij} \in E_H(M)$, for $i, j = 1, 2$, such that

$$B_L(e_i \phi m_1, e_j \phi m_2) = B(\alpha_{ij} m_1, m_2), \quad \text{for all } m_1, m_2 \in M.$$

As $e = e_1 + e_2$ and ϕ is an isometry, we get

$$\sum_{i,j=1,2} \text{tr}_H^G(\alpha_{ij}) = \theta.$$

Now it can be shown that $\alpha_{ij}^\sigma = \alpha_{ji}$. So $\text{tr}_H^G(\alpha_{12}) + \text{tr}_H^G(\alpha_{21}) = \text{tr}_H^G(\alpha_{12}) + \text{tr}_H^G(\alpha_{12})^\sigma$. This sum belongs to $J(E_G(M))$, as $E_G(M)$ is a local ring and $\text{char}(k) = 2$. So $\theta_{ii} := \text{tr}_H^G(\alpha_{ii})$ is a unit in $E_G(M)$ for some i . Lemma 4.3 gives an isometry $(M, B_{\theta_{ii}}) \rightarrow \text{Ind}_H^G(\alpha_{ii}M, \hat{B}_{\alpha_{ii}})$. But $\dim(\alpha_{ii}M) < \dim(L)$, contradicting our choice of L . This establishes our claim and completes the proof. \square

5. SYMMETRIC VERTICES

For the rest of the paper k is an algebraically closed field of characteristic 2.

5.1. Projective modules and involutions. We interpret some of the results of [8] and [22] in terms of induction of forms. The structural map of the left regular module kG is $\ell : kG \rightarrow E(kG)$, where $\ell(x)y := xy$, for $x, y \in kG$. We also consider the k -algebra isomorphism $r : kG^{\text{op}} \rightarrow E_G(kG)$, defined by $r(x)y := yx$. Also kG is a left $k(G \times G)$ -module via $(g_1, g_2)x := g_1xg_2^{-1}$, for $g_1, g_2 \in G$ and $x \in kG$.

Recall that B_1 is the $G \times G$ -form on kG such that for all $g_1, g_2 \in G$ in kG

$$B_1(g_1, g_2) = \begin{cases} 1_k, & \text{if } g_1 = g_2, \\ 0_k, & \text{if } g_1 \neq g_2. \end{cases}$$

We will use σ to denote the adjoint of B_1 on $E(kG)$. Then $\ell(x)^\sigma = \ell(x^\circ)$ and $r(x)^\sigma = r(x^\circ)$, for all $x \in kG$, where $^\circ$ is the contragredient map on kG .

Recall from 3.2 that the G -invariant bilinear forms on kG are $B_{r(a)}$, for $a \in kG$. We simplify $B_{r(a)}$ to B_a . So $B_a(x, y) = B_1(xa, y)$, for all $x, y \in kG$. Then B_a is non-degenerate if a is a unit in kG , symmetric if $a = a^\circ$ and symplectic if $a = a^\circ$ and $B_1(a, 1) = 0$. As an important example, B_t is a non-degenerate symplectic form, for each involution $t \in G$.

Now suppose that a is a symmetric unit in kG cf. [19]. So $a = a^\circ$. Set σ_a as the adjoint of B_a on $E(M)$. Then it is easy to check that

$$(3) \quad \ell(x)^{\sigma_a} = \ell(x^\circ) \quad \text{and} \quad r(x)^{\sigma_a} = r(ax^\circ a^{-1}), \quad \text{for all } x \in kG.$$

The fact that the adjoint has a different effect on the isomorphic subrings $\ell(kG)$ and $r(kG)$ of $E(kG)$ has been neglected in previous works such as [18] and [8].

Let P be a self-dual principal indecomposable kG -module. If P is the projective cover of the trivial module then it affords a diagonalizable form and at least one symplectic form. Otherwise P has symmetric type if and only if it has symplectic type. In fact, each non-degenerate G -invariant symplectic form on P is the polarization of a G -invariant quadratic form. For full details, see [8].

Our first result includes *Fong's Lemma*:

Lemma 5.1. *Let M be a non-trivial self-dual irreducible kG -module. Then M affords a unique, up to scalars, non-degenerate G -invariant symplectic bilinear form.*

The non-symplectic form B_1 is degenerate on each direct summand of kG which is isomorphic to $P(M)$.

Proof. There is a point ϵ of $E_G(M)$ such that $P(M) \cong kGe$ for $e \in \epsilon$. Then $\pi_\epsilon : kG \rightarrow E(M)$ is surjective. As $P(M) \cong P(M)^*$, we have $\ker(\epsilon)^\sigma = \ker(\epsilon)$. So σ induces an involution, also denoted σ , on $E(M)$. Let B be the non-degenerate

symmetric form on M whose adjoint is σ , as given by Lemma 3.1. Then B is G -invariant as B_1 is G -invariant, symplectic as M has no trivial submodule, and unique up to multiplication by a non-zero scalar, as $E_G(M) \cong k$.

We claim that B_1 is degenerate on kGe , for all $e \in \epsilon$. Otherwise Lemma 3.5 shows that $r(e)^\sigma = r(e)$ for some $e \in \epsilon$. But then $\ell(e)^\sigma = \ell(e)$. So $\pi_\epsilon(e)$ is an σ -invariant primitive idempotent in $E(M)$. Again by Lemma 3.5, B is non-degenerate on the 1-dimensional space $\pi_\epsilon(e)M$. This contradicts the fact that B is symplectic and our claim follows. \square

Our next result includes a proof of [8, (1.6)].

Lemma 5.2. *B_1 restricts to a diagonalizable G -form on each direct summand of kG which is isomorphic to $P(k_G)$. Also $\dim P(k_G)/|G|_2$ is odd.*

Proof. We may write $kG \cong \sum P(S)^{\dim S}$ where S ranges over all irreducible kG -modules. As $P(S) \cong P(k_G)^*$ if and only if $S \cong k_G$, Lemma 3.8 implies that B_1 is non-degenerate on each direct summand of kG which is isomorphic to $P(k_G)$. Now B_1 is symplectic on each direct summand of kG which is not isomorphic to $P(k_G)$. As B_1 is not symplectic on kG , it is not symplectic on any direct summand isomorphic to $P(k_G)$.

Now $|G|_{2'} = \sum (\dim P(S)/|G|_2) \dim S$, where each $\dim P(S)/|G|_2$ is an integer. If $S \cong S^*$ and $S \not\cong k_G$ then $\dim S$ is even, by Fong's Lemma. If $S \not\cong S^*$, then S and S^* contribute equally to the sum. Thus $|G|_{2'} \equiv \dim P(k_G)/|G|_2 \pmod{2}$. \square

Recall that $E(kG)$ is a G -algebra, as kG is a kG -module. So by definition ${}^g f(z) := gf^{-1}(z)$, for all $g \in G, f \in E(kG)$ and $z \in kG$. To describe the relative trace maps $\text{tr}_H^G : E_H(kG) \rightarrow E_G(kG)$, we first need a description of $E(kG)$. We use B_1 to identify $kG \otimes kG$ with $E(kG)$: $x \otimes y$ is the rank-1 endomorphism

$$(4) \quad (x \otimes y)(z) = B_1(y, z)x, \quad \text{for } z \in kG.$$

Then the structure of the involutory G -algebra $(E(kG), \sigma)$ is given by

$$(5) \quad (x \otimes y)^\sigma = y \otimes x, \quad \text{and} \quad {}^g(x \otimes y) = gx \otimes gy, \quad \text{for } g \in G.$$

Using this we see that

$$\text{tr}_1^G(x \otimes y) = r(y^\sigma x), \quad \text{for all } x, y \in kG.$$

It is useful to list the elements of G as

$$1, t_1, \dots, t_m, \quad \begin{matrix} g_1, & \dots, & g_n \\ g_1^{-1}, & \dots, & g_n^{-1} \end{matrix}$$

where each t_i is an involution.

Lemma 5.3. *A basis for the σ -invariant elements in $E_G(kG)$ is*

$$r(1), r(t_1), \dots, r(t_m), r(g_1 + g_1^{-1}), \dots, r(g_n + g_n^{-1}).$$

Of these $r(1)$ and $r(g_j + g_j^{-1})$ are $(1, \sigma)$ -projective, while $r(t_i)$ is (H, σ) -projective, for $H \leq G$, if and only if ${}^g t_i \in H$, for some $g \in G$.

Proof. The first statement follows from $kG^{op} \cong E_G(kG)$ and (3). Let $g \in G$. Then $1 \otimes 1$ and $g \otimes 1 + 1 \otimes g$ are σ -invariant and

$$\text{tr}_1^G(1 \otimes 1) = r(1), \quad \text{tr}_1^G(g \otimes 1 + 1 \otimes g) = r(g + g^{-1}).$$

So $r(1)$ and $r(g + g^{-1})$, for $g \neq g^{-1}$, are $(1, \sigma)$ -projective.

Let $t = t_i$. Then $t \otimes 1 + 1 \otimes t \in E_{\langle t \rangle}(M)$ is σ -invariant and

$$\mathrm{tr}_{\langle t \rangle}^G(t \otimes 1 + 1 \otimes t) = r(t).$$

So $r(t)$ is $(\langle t \rangle, \sigma)$ -projective, if $t \neq 1$.

Let H be a subgroup of G . Then the endomorphisms $\mathrm{tr}_1^H(a \otimes b)$ span $E_H(M)$, as a, b range over all elements of G . Now

$$\mathrm{tr}_1^H(a \otimes b)(g) = \begin{cases} gb^{-1}a, & \text{if } g \in Hb. \\ 0, & \text{if } g \in G \setminus Hb. \end{cases}$$

So $\mathrm{tr}_1^H(a \otimes b)$ is σ -invariant if and only if $Ha = Hb$ and $b^{-1}a = a^{-1}b$ i.e. if and only if $t := b^{-1}a$ is an involution such that ${}^bt \in H$. The last statement of the lemma follows from this. \square

Lemma 5.4. *Let $H \leq G$ and let a be a unit in kH . Then*

$$\mathrm{Ind}_H^G(kH, B_a) \cong (kG, B_a).$$

In particular $(kG, B_t) \cong \mathrm{Ind}_{\langle t \rangle}^G(k\langle t \rangle, B_t)$, for all $t \in G$ with $t^2 = 1$.

Proof. Let $r_H(a)$ be the endomorphism $x \rightarrow xa$ of kH . Then $r_H(a)$ extends to a kH -endomorphism of kG (acting as 0 on $k(G \setminus H)$) and $\mathrm{tr}_H^G(r_H(a)) = r(a)$. The Lemma is a consequence of this fact. \square

Lemma 5.5. *Two involutions $s, t \in G$ are G -conjugate if and only if $(k\langle s \rangle, B_s)$ is a component of $\mathrm{Res}_{\langle s \rangle}^G(kG, B_t)$.*

Proof. It is clear that there is an $\langle s \rangle$ -isometry $(k\langle s \rangle, B_s) \rightarrow \mathrm{Res}_{\langle s \rangle}^G(kG, B_t)$ if and only if $B_t(x, sx) \neq 0$, for some $x \in kG$. If $x = \sum_{g \in G} x_g g$, with $x_g \in k$, then

$$B_t(x, sx) = B_1(xt, sx) = \sum_{g \in G} x_g x_{sgt} = \sum_{g \in G, g = sgt} x_g^2,$$

using $x_g x_{sgt} + x_{sgt} x_g = 0$. So if $B_t(x, sx) \neq 0$ then $g = sgt$, for some $g \in G$. In that case $s = gtg^{-1}$ is conjugate to t .

Conversely, if $s = gtg^{-1}$, then $B_t(g, sg) = 1$. \square

Our next result is an elaboration of parts of [8, Section 3].

Lemma 5.6. *Let e be a primitive idempotent in kG and let \hat{B} be a non-degenerate G -invariant symplectic form on kGe . Then there is an involution $t \in G$ such that B_t is non-degenerate on kGe and $(k\langle t \rangle, B_t)$ is a component of $\mathrm{Res}_{\langle t \rangle}^G(kGe, \hat{B})$.*

Proof. By Lemma 3.6 there is $a \in ekGe^\sigma$ so that $\hat{B}(xe, ye) = B_a(x, y)$, for all $x, y \in kG$. Then $a = a^\sigma$ and $B_1(a, 1) = 0$, as \hat{B} is symplectic. Write $a = \sum_{i=1}^m \alpha_i t_i + \sum_{j=1}^n \beta_j (g_j + g_j^{-1})$, with $\alpha_i, \beta_j \in k$. Now $E_G(kGe)$ is a local ring. So each $B_{\beta_j(g_j + g_j^{-1})}$ is degenerate on kGe . It follows that $B_{\alpha_i t_i}$ is non-degenerate on kGe , for some i . Set $t = t_i$. Then B_t is non-degenerate on kGe and $B_a(e, te) = \alpha_i \neq 0$. So $ke + kte$ is a B_a -direct summand of $\mathrm{Res}_{\langle t \rangle}^G(kGe)$ which is isomorphic to $k\langle t \rangle$. We conclude that $(k\langle t \rangle, B_t)$ is a component of $\mathrm{Res}_{\langle t \rangle}^G(kGe, \hat{B})$. \square

We note that B_t is non-degenerate on kGe if and only if $kGe = kGf$ for some primitive idempotent $f \in kG$ with $tft = f^\sigma$, using Lemma 3.5 and (3). This is [8, 3.4 and 3.5]. Our last result strengthens [22, 6.5].

Corollary 5.7. *Suppose that B_t is non-degenerate on kGe where $t \in G$ and $t^2 = 1$. Then (kGe, B_t) is a component of $\text{Ind}_{\langle t \rangle}^G \text{Res}_{\langle t \rangle}^G(kGe, B_t)$.*

Let M be the irreducible head of kGe . Then $\text{Res}_{C_G(t)}^G M$ has diagonalizable type. In particular $k_{C_G(t)}$ is a submodule and a quotient module of $\text{Res}_{C_G(t)}^G M$.

Proof. If $t = 1$, then $kGe \cong P(k_G)$, and Lemma 5.2 gives all conclusions. So assume that t is an involution. By Lemma 5.6 there is an involution $s \in G$ and an isometry $(k\langle s \rangle, B_s) \rightarrow \text{Res}_{\langle s \rangle}^G(kGe, B_t)$. Then s and t are G -conjugate, according to Lemma 5.5. This proves the first assertion.

We may assume that $M \not\cong k_G$. Let B be the G -invariant symplectic form on M . Recall that $\pi_\epsilon : kG \rightarrow E(M)$ has kernel \mathfrak{M}_ϵ . Now σ_t is an involution on kG which fixes \mathfrak{M}_ϵ . So it induces an involution σ_t on $E(M)$. As $\ell(g)^{\sigma_t} = \ell(tg^{-1}t)$, we get $\pi_\epsilon(g)^{\sigma_t} = \pi_\epsilon(tg^{-1}t)$. So the corresponding non-degenerate symmetric bilinear form on M is $\hat{B}(m_1, m_2) := B(tm_1, m_2)$, for all $m_1, m_2 \in M$. Then \hat{B} is $C_G(t)$ -invariant.

Let $f \in kG$ be the σ_t -invariant primitive idempotent with $kGe = kGf$, as given by Lemma 3.5. Then $\pi_\epsilon(f)$ is a σ_t -invariant primitive idempotent in $E(M)$. It follows that \hat{B} is a diagonalizable form on M . The last assertion now follows from Lemma 2.4. \square

Finally we prove Theorem 1.6 of the introduction. This gives an intriguing new criterion for a principal indecomposable module to be of quadratic type, which depends entirely on the irreducible head of the module.

Proposition 5.8. *Let M be a non-trivial self-dual irreducible kG -module, with symplectic form B , and let $t \in G$ be an involution. Then $P(M)$ is a B_t -component of kG if and only if $B(tm, m) \neq 0$, for some $m \in M$.*

Proof. We proved the ‘only if’ in Corollary 5.7. So suppose that there is $m \in M$ such that $B(tm, m) \neq 0$. Define \hat{B} on M as above, with an involution σ_t on $E(M)$ which lifts to the involution σ_t of kG . Now \hat{B} is non-degenerate on the subspace km of M . So by Lemma 3.5 there is a unique σ_t -invariant primitive idempotent $\hat{f} \in E(M)$ such that $\hat{f}(M) = km$. By Lemma 2.3(iii) there is a σ_t -invariant primitive idempotent $f \in kG$ such that $\pi_\epsilon(f) = \hat{f}$. In particular $kGf \cong P(M)$. Lemma 3.5 guarantees that B_t is non-degenerate on kGf . \square

Example 5.9. [14, 11.2] *Let S_n be the symmetric group of degree $n \geq 2$ and let λ be a non-trivial 2-regular partition of n . The Specht module S^λ affords a non-zero S_n -invariant symplectic bilinear form B such that $D^\lambda := S^\lambda / (S^\lambda)^\perp$ is an irreducible kS_n -module. Let T be a λ -tableau, so there is a corresponding polytabloid e_T in S^λ . Now T is a filling of the boxes in a Young diagram for λ with the symbols $1, 2, \dots, n$. Let t be the involution in S_n which reverses the entries in each row. Then the tabloid labelled by T is the only tabloid common to te_T and e_T . So $B(te_T, e_T) = 1_k$. It follows that $P(D^\lambda)$ is a B_t -component of kS_n . Note that if λ has ℓ non-zero parts then t is a product of $(n - \ell)/2$ commuting transpositions.*

5.2. Puig correspondence. In parts 5.2 through to 5.6 we take M to be an indecomposable kG -module, with Green vertex V and V -source Z .

We will use the following notation:

- Ω is the point of $E_G(\text{Ind}_V^G Z)$ such that $\omega \text{Ind}_V^G Z \cong M$, for all $\omega \in \Omega$.
- $\mathcal{F} : E(M) \rightarrow E(\text{Ind}_V^G Z)$ is the embedding of G -algebras induced by Ω .

- δ is the point of $E_V(M)$ such that $dM \cong Z$, for all $d \in \delta$.
- Δ is the point of $E_V(\text{Ind}_V^G Z)$ containing $\mathcal{F}(\delta)$.

Let $N_G(V, Z)$ be the stabilizer of Z in $N_G(V)$. Set $N := N_G(V, Z)/V$. Then $E(P_\Delta)$ is an N -algebra. So P_Δ is a module for a twisted group algebra $k_\gamma N$ of N over k . Likewise $E(P_\delta)$ is an N -algebra and P_δ is a module for a twisted group algebra $k_{\gamma'} N$ of N over k . According to [28, (26.1)], P_Δ is the regular $k_\gamma N$ -module. Now \mathcal{F} induces an embedding of N -algebras $E(P_\delta) \rightarrow E(P_\Delta)$. This in turn induces a group isomorphism between the central extensions of N corresponding to the cocycles γ and γ' . In this way, P_δ can and will be identified with an indecomposable component of P_Δ , as $k_\gamma N$ -modules.

Lemma 5.10. *Suppose that $Z \cong Z^*$ and either Z or M is of symmetric type. Then $k_\gamma N \cong kN$. So P_Δ is the regular kN -module.*

Proof. Suppose first that Z has a symmetric V -form B_0 . Let σ_0 be the adjoint of B_0^G on $E(\text{Ind}_V^G Z)$. As $Z \cong Z^*$, Lemma 3.4 implies that $\Delta^{\sigma_0} = \Delta$. So $\mathfrak{M}_\Delta^{\sigma_0} = \mathfrak{M}_\Delta$ and σ_0 is an involution on $E_V(\text{Ind}_V^G Z)/\mathfrak{M}_\Delta$. In this way $(E(P_\Delta), \sigma_0)$ is a simple involutory N -algebra.

By Lemma 3.9, there is a symmetric form B_{σ_0} on P_Δ such that the action of N on $E(P_\Delta)$ lifts to a representation $N \rightarrow \text{GL}(P_\Delta, B_{\sigma_0})$. In particular $k_\gamma N \cong kN$ as twisted group algebras and P_Δ is the regular kN -module.

Conversely suppose that M has a symmetric G -form B . Let σ be the adjoint of B on $E(M)$. Then $\delta^\sigma = \delta$. So $\mathfrak{M}_\delta^\sigma = \mathfrak{M}_\delta$ and σ induces an involution on $E_V(M)/\mathfrak{M}_\delta$. According to Lemma 3.9, the action of N on $E(P_\delta)$ lifts to a representation $N \rightarrow \text{GL}(P_\delta, B_\sigma)$, where B_σ is a symmetric form on P_δ . Thus $k_{\gamma'} N \cong kN$ as twisted group algebras. But $k_\gamma N \cong k_{\gamma'} N$. So as before P_Δ is the regular kN -module. \square

Set $N^* = N_G^*(V, Z)/V$, where $N_G^*(V, Z)$ is the stabilizer of $\{Z, Z^*\}$ in $N_G(V)$. So $[N^* : N] \leq 2$. The following is based on [28, (14.8)].

Lemma 5.11. *Let L be a component of $\text{Ind}_V^G Z$ and let ϵ be a point of $E_V(L)$ contained in Δ . For $V \leq H \leq G$ set $N_H = N_H(V, Z)/V$ and $N_H^* = N_H^*(V, Z)/V$. Then for all $f \in E_V(L)\epsilon E_V(L)$ we have:*

- (i) $\pi_\epsilon \text{tr}_V^H(f) = \text{tr}_1^{N_H} \pi_\epsilon(f)$ and $\pi_\epsilon \text{res}_V^H : E_H(L) \rightarrow E_{N_H}(P_\epsilon)$ is onto.
- (ii) If σ is a G -involution of $E(L)$, then $\pi_{\epsilon, \epsilon^\sigma} \text{tr}_V^H(f) = \text{tr}_1^{N_H^*} \pi_{\epsilon, \epsilon^\sigma}(f)$ and $\pi_{\epsilon, \epsilon^\sigma} \text{res}_V^H : E_H(L) \rightarrow (E(P_\epsilon) \times E(P_{\epsilon^\sigma}))^{N_H^*}$ is onto.

Proof. (i) follows from Remark (19.9) in [28] as $E_H(L) = \text{tr}_V^H(E_V(L))$.

From the proof of [28, (14.7)] we see that $\pi_{\epsilon, \epsilon^\sigma} \text{tr}_V^H(f) = \text{tr}_1^{N_H^*}(\pi_{\epsilon, \epsilon^\sigma}(f))$. So $\pi_{\epsilon, \epsilon^\sigma}$ maps $\text{tr}_V^H(E_V(L))$ onto $\text{tr}_1^{N_H^*}(E(P_\epsilon) \times E(P_{\epsilon^\sigma}))$. This is a 2-sided ideal of the fixed point subalgebra $(E(P_\epsilon) \times E(P_{\epsilon^\sigma}))^{N_H^*}$ which is contained in $\pi_{\epsilon, \epsilon^\sigma} E_H(L)$.

We now modify [28, (14.8)]. As $L \mid \text{Ind}_V^G Z$, we have $1_L = \text{tr}_V^H(\iota)$ for some $\iota \in E_V(L)\epsilon E_V(L)$. So $1_{P_\epsilon + P_{\epsilon^\sigma}} = \text{tr}_1^{N_H^*} \pi_{\epsilon, \epsilon^\sigma}(\iota)$ and hence

$$\text{tr}_1^{N_H^*}(E(P_\epsilon) \times E(P_{\epsilon^\sigma})) = (E(P_\epsilon) \times E(P_{\epsilon^\sigma}))^{N_H^*}.$$

Now (ii) follows from this and the previous paragraph. \square

In our situation the Puig correspondence [28, (19.1)] is a multiplicity preserving bijection between the indecomposable components of $\text{Ind}_V^G Z$ with vertex V and

the indecomposable components of P_Δ . More concretely, if e is a primitive idempotent in $E_G(\text{Ind}_V^G Z)$ such that $e \text{Ind}_V^G Z$ has vertex V , then $\pi_\Delta(e)$ is a primitive idempotent in $E_N(P_\Delta)$, and $\pi_\Delta(e)P_\Delta$ is the Puig correspondent of $e \text{Ind}_V^G Z$.

5.3. Theorem 1.2(i).

Proposition 5.12. *Suppose that Z affords a non-degenerate V -invariant symmetric bilinear form B_0 . Then the Puig correspondent of $P(k_N)$ is the unique indecomposable B_0^G -component of $\text{Ind}_V^G Z$ that has vertex V .*

Proof. Lemma 5.10 applies, and we adopt its notation. Let $d \in \Delta$ be the orthogonal projection $\text{Ind}_V^G(Z) \rightarrow 1 \otimes Z$. So $d^{\sigma_0} = d$ and $\text{tr}_V^G(d) = 1_{\text{Ind}_V^G Z}$. Set $\bar{d} := \pi_\Delta(d)$. Then $1_{P_\Delta} = \text{tr}_1^N(\bar{d})$, using Lemma 5.11(i). Moreover $\bar{d}^{\sigma_0} = \bar{d}$. So $(P_\Delta, B_{\sigma_0}) \mid \text{Ind}_1^N(\bar{d}P_\Delta, \hat{B}_{\bar{d}})$, according to Lemma 4.3. But $\dim(\bar{d}P_\Delta) = 1$. So $(\bar{d}P_\Delta, \hat{B}_{\bar{d}}) \cong (k_1, B_1)$ and thus $\text{Ind}_1^N(\bar{d}P_\Delta, \hat{B}_{\bar{d}}) \cong (kN, B_1)$. So we can and do identify (P_Δ, B_{σ_0}) with (kN, B_1) .

Write $\text{Ind}_V^G(Z) = L_1 \dot{+} L_2 \cdots + \cdots \dot{+} L_n$, where the L_i are indecomposable kG -modules and L_1 is the Puig correspondent of $P(k_N)$. Then $L_i \not\cong L_1^*$ for $i > 1$. So B_0^G is non-degenerate on L_1 , by Lemma 3.8.

Now suppose that B_0^G is non-degenerate on L_i , where L_i has vertex V . Then L_i has V -source Z . Let $\omega \in E_G(\text{Ind}_V^G Z)$ be orthogonal projection onto L_i . Then $\pi_\Delta(\omega)$ is a σ_0 -invariant primitive idempotent in $E_N(P_\Delta)$. So $\pi_\Delta(\omega)kN \cong P(k_N)$, by Lemma 5.1. We deduce that $L_i = L_1$. \square

From now on we assume that M is of symmetric type.

Proposition 5.13. *The following are equivalent:*

- (i) *Each symmetric vertex of M is a Green vertex of M .*
- (ii) *Z has symmetric type and M is the Puig correspondent of $P(k_N)$.*
- (iii) *Z has symmetric type and if B_0 is a V -form on Z then M is a B_0^G -component of $\text{Ind}_V^G Z$.*
- (iv) *M has G -form B such that Z is a B -component of $\text{Res}_V^G M$.*

Proof. Assume (i). Then by Lemma 4.7, there is symmetric kV -module (Y, B_1) such that Y is indecomposable and M is a B_1^G -component of $\text{Ind}_V^G(Y)$. Then Y is a V -source of M . But $Z = {}^n Y$, for some $n \in N_G(V)$. So (iii) holds. Moreover, (i) and (ii) are equivalent, by Proposition 5.12.

Lemma 5.10 applies if (ii), (iii) or (iv) hold. We adopt its notation.

Assume (iii). Then M has a V -projective G -form. So (i) is true. Lemma 3.5 implies that there is a σ_0 -invariant idempotent ω in Ω and B_0^G is non-degenerate on $\omega \text{Ind}_V^G Z \cong M$. Set $\bar{\omega} = \pi_\Delta(\omega)$. Then $\bar{\omega}kN \cong P(k_N)$. Lemma 5.2 implies that B_1 is non-degenerate on a 1-dimensional subspace of $\bar{\omega}kN$. So again using Lemma 3.5, there is a σ_0 -invariant primitive idempotent $\bar{d} \in \bar{\omega}E(kN)\bar{\omega}$.

Now π_Δ restricts to a surjective map $\omega E_V(\text{Ind}_V^G Z)\omega \rightarrow \bar{\omega}E(kN)\bar{\omega}$. So by Lemma 2.3(iii) there is a σ_0 -invariant primitive idempotent $d \in \omega E_V(\text{Ind}_V^G Z)\omega$ such that $\pi_\Delta(d) = \bar{d}$. In particular $d \in \Delta$ and $d = \omega d \omega$. Then B_0^G is non-degenerate on the V -component $d \text{Ind}_V^G Z$ of $\omega \text{Ind}_V^G(Z) \cong M$. So (iv) holds.

Assume (iv). By Lemma 3.5 this means that there is $d \in \delta$ with $d^\sigma = d$. As $\pi_\delta(d)$ is a σ -invariant primitive idempotent in $E(P_\delta)$, B_σ is a diagonalizable N -form on P_δ . So $P_\delta \cong P(k_N)$, using Lemma 5.1. Now $P(k_N)$ is the only self-dual principal

indecomposable kN -module which has multiplicity 1 in kN . So $P_\delta \cong P(k_N)$, when regarded as a component of P_Δ . So (ii) holds. \square

The principal 2-block $\mathcal{B}_0 = \mathcal{B}_0(G)$ of G is the block of kG which contains k_G .

Corollary 5.14. *If some Green vertex of M is a symmetric vertex of M then M belongs to the principal 2-block of G .*

Proof. From Proposition 5.13, M is the Puig correspondent of $P(k_N)$. Set $C = VC_G(V)$. Then $\text{Res}_{C/V}^N(P(k_N)) \cong P(k_{C/V})^m$, for some $m \geq 1$. So $(V, P(k_{C/V}))$ is a root of M , in the terminology of [16]. Now $P(k_{C/V})$ is in $\mathcal{B}_0(C)$, and Brauer's Third Main Theorem implies that $(V, \mathcal{B}_0(C))$ is a \mathcal{B}_0 -subpair. So M belongs to \mathcal{B}_0 , according to [16]. \square

5.4. Theorem 1.2(ii).

Proposition 5.15. *Suppose that the sources of M are self-dual, but B is degenerate on each direct summand of $\text{Res}_V^G M$ that is a source of M . Then*

- (i) *There is $V \leq T \leq N_G(V, Z)$ with $[T : V] = 2$ such that $\text{Ind}_V^T Z$ is a B -component of $\text{Res}_T^G M$.*
- (ii) *Let B_0 be a T -form on $\text{Ind}_V^T Z$ such that Z is not a B_0 -component of $\text{Res}_V^T \text{Ind}_V^T Z$. Then M is a B_0^G -component of $\text{Ind}_V^G Z$.*
- (iii) *Either V or T is a symmetric vertex of M .*

Proof. Lemma 5.10 applies, and we adopt its notation.

As B is degenerate on each direct summand of $\text{Res}_V^G M$ that is isomorphic to Z , σ does not fix any idempotent in δ . So by Lemma 2.3(i), σ does not fix any primitive idempotent in $E(P_\delta)$. This means that (P_δ, B_σ) is a symplectic kN -module. Lemma 5.6 gives an involution $t \in N$ such that B_t is non-degenerate on P_δ . Moreover $\text{Res}_{\langle t \rangle}^G(P_\delta, B_\sigma)$ has a component $(k\langle t \rangle, B_t)$. So there is a σ -invariant primitive idempotent $\bar{y} \in E_{\langle t \rangle}(P_\delta)$.

Let $T \leq N$ with $T/V = \langle t \rangle$. Then $\pi_\delta \text{res}_V^T : E_T(M) \rightarrow E_{\langle t \rangle}(P_\delta)$ is surjective, by Lemma 5.11(i). So by Lemma 2.3(iii), there is a primitive σ -invariant idempotent $y \in E_T(M)$ with $\pi_\delta(y) = \bar{y}$. So $Y = yM$ is a B -direct summand of $\text{Res}_T^G M$. Now Y is V -projective, $|T/V| = 2$ and Z is a component of Y_V . So $Y \cong \text{Ind}_V^T Z$. The conclusion of (i) follows.

Assume the hypothesis of (ii) and set $Y := \text{Ind}_V^T Z$. Note that $\text{Ind}_T^G Y \cong \text{Ind}_V^G Z$. Let σ_0 be the adjoint of B_0^G on $E(\text{Ind}_V^G Z)$. Now $(E(P_\Delta), \sigma_0)$ is an involutory N -algebra. So $k_\gamma N \cong kN$, P_Δ is the regular kN -module and σ_0 is the adjoint of a symmetric N -form B_{σ_0} on P_Δ . By hypothesis on B_0 , the N -form B_{σ_0} is symplectic.

Let $e \in E_T(\text{Ind}_V^G Z)$ be orthogonal projection onto $1 \otimes Y$. Then $\text{tr}_T^G(e) = 1_{\text{Ind}_V^G Z}$ and $e \in \text{tr}_V^T(E_V(\text{Ind}_V^G Z) \Delta E_V(\text{Ind}_V^G Z))$. Set $\bar{e} = \pi_\Delta(e)$. Then

$$1_{E(P_\Delta)} = \pi \text{tr}_T^G(e) = \text{tr}_{\langle t \rangle}^N(\bar{e}), \quad \text{using Lemma 5.11(i).}$$

As $\bar{e}^{\sigma_0} = \bar{e}$, Lemma 4.3 gives a kN -isometry $(P_\Delta, B_{\sigma_0}) \rightarrow \text{Ind}_{\langle t \rangle}^N(\bar{e}P_\Delta, \hat{B}_{\bar{e}})$. But $\dim(\bar{e}P_\Delta) = 2$. So this isometry is surjective, as both sides have dimension $|N|$. Now B_{σ_0} is symplectic, and $\langle t \rangle$ is cyclic of order 2. So $(\bar{e}P_\Delta, \hat{B}_{\bar{e}}) \cong (k\langle t \rangle, B_t)$. We deduce that $(P_\Delta, B_{\sigma_0}) \cong (kN, B_t)$.

Now P_δ is a B_t -component of P_Δ . So there is a primitive σ_0 -invariant idempotent $\bar{\omega} \in E_N(P_\Delta)$ with $\bar{\omega}P_\Delta \cong P_\delta$. Since $\pi_\Delta \text{res}_V^G : E_G(\text{Ind}_V^G Z) \rightarrow E_N(P_\Delta)$ is surjective,

Lemma 2.3(iii) gives a primitive σ_0 -invariant idempotent $\omega \in E_G(\text{Ind}_V^G Z)$ with $\pi_\Delta(\omega) = \overline{\omega}$. So $\omega \text{Ind}_V^G Z$ is a B_0^G -direct summand of $\text{Ind}_V^G Z$. But $\omega \text{Ind}_V^G(Z) \cong M$. The conclusion of (ii) follows.

(iii) holds as M has a T -projective symmetric G -form. \square

5.5. Theorem 1.2(iii).

Proposition 5.16. *Suppose that no source of M is self-dual. Then*

- (i) $N_G^*(V, Z)$ has a subgroup T which contains V such that $[T : V] = 2$ and $\text{Ind}_V^T Z$ is a B -component of $\text{Res}_T^G M$. Then $N^* = N \rtimes T/V$.
- (ii) M is a B_0^G -component of $\text{Ind}_V^G Z$, if B_0 is a symmetric T -form on $\text{Ind}_V^T Z$.
- (iii) T is a symmetric vertex of M .

Proof. Set Δ^σ as the point of $E_G(\text{Ind}_V^G Z)$ corresponding to Z^* . Lemma 3.4 implies that $\delta^\sigma \neq \delta$. So $(E(P_\delta) \times E(P_{\delta^\sigma}), \sigma)$ is an involutory N^* -algebra satisfying the hypothesis of Lemma 3.1(ii). Moreover, this algebra is embedded in the N^* -algebra $E(P_\Delta) \times E(P_{\Delta^\sigma})$ as follows. According to Lemma 5.11(ii), the restriction map $\pi_{\Delta, \Delta^\sigma} : E_G(\text{Ind}_V^G Z) \rightarrow (E(P_\Delta) \times E(P_{\Delta^\sigma}))^{N^*}$ is surjective. Let $\omega \in \Omega$. Then $\overline{\omega} := \pi_{\Delta, \Delta^\sigma}(\omega)$ is a primitive idempotent in $(E(P_\Delta) \times E(P_{\Delta^\sigma}))^{N^*}$. We identify $E(P_\delta) \times E(P_{\delta^\sigma})$ with $\overline{\omega}(E(P_\Delta) \times E(P_{\Delta^\sigma}))\overline{\omega}$, and $P_\delta + P_{\delta^\sigma}$ with $\overline{\omega}(P_\Delta + P_{\Delta^\sigma})$.

By Lemma 3.10 there is a commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & O & \xrightarrow{\text{inc}} & H & \xrightarrow{\theta} & N^* & \longrightarrow & 1 \\
 & & \downarrow \eta & & \downarrow \chi & & \downarrow \rho & & \\
 1 & \longrightarrow & K & \xrightarrow{\text{inc}} & \text{Sp}(P_\delta, P_{\delta^\sigma}) & \xrightarrow{\rho} & \text{PGL}(P_\delta, P_{\delta^\sigma}, \sigma) & \longrightarrow & 1
 \end{array}$$

where O is a finite cyclic group of odd order and $\theta(C_H(O)) = N$. Each element of $H \setminus C_H(O)$ maps P_δ onto P_{δ^σ} . Moreover σ is the adjoint of a symplectic H -form B_σ on $P_\delta + P_{\delta^\sigma}$, with

$$(6) \quad (P_\delta)^\perp = P_\delta \quad \text{and} \quad (P_{\delta^\sigma})^\perp = P_{\delta^\sigma}.$$

Now $e_\eta = \frac{1}{|O|} \sum_{\lambda \in O} \eta(\lambda^{-1})\lambda$ is a central idempotent in kH such that $P_\Delta + P_{\Delta^\sigma} \cong kHe_\eta$ as kH -modules. So $E_{N^*}(P_\Delta + P_{\Delta^\sigma}) \cong e_\eta kHe_\eta$.

By Lemma 5.6 there is an involution $t \in H$ such B_t is non-degenerate on $P_\delta + P_{\delta^\sigma}$ and $k\langle t \rangle$ is a B_σ -component of $\text{Res}_{\langle t \rangle}^H(P_\delta + P_{\delta^\sigma})$. This means that there is $p \in P_\delta + P_{\delta^\sigma}$ such that $B_\sigma(p, tp) \neq 0$. Write $p = p_1 + p_2$ where $p_1 \in P_\delta$ and $p_2 \in P_{\delta^\sigma}$.

We claim that $t \notin C_H(O)$. Otherwise, $tp_1 \in P_\delta$ and $tp_2 \in P_{\delta^\sigma}$. Then

$$\begin{aligned}
 B_\sigma(p, tp) &= B_\sigma(p_1, tp_2) + B_\sigma(p_2, tp_1), & \text{by (6)} \\
 &= B_\sigma(p_1, tp_2) + B_\sigma(tp_1, p_2), & \text{as } B_\sigma \text{ is symmetric} \\
 &= 0, & \text{as } t^\sigma = t.
 \end{aligned}$$

This contradicts our choice of p and thus establishes our claim.

Now $tp_1 \in P_{\delta^\sigma}$ and $tp_2 \in P_\delta$. So $B_\sigma(p, tp) = B_\sigma(p_1, tp_1) + B_\sigma(p_2, tp_2)$. Replace p by p_1 or tp_2 so that $p \in P_\delta$ and $B_\sigma(p, tp) \neq 0$. Then replace p by $\sqrt{B_\sigma(p, tp)^{-1}}p$, so that $B_\sigma(p, tp) = 1$.

Define $\overline{y} \in E_{\langle t \rangle}(P_\delta + P_{\delta^\sigma})$ by $\overline{y}(x) = B(x, tp)p + B(x, p)tp$, for all $x \in P_\delta + P_{\delta^\sigma}$. Then \overline{y} is orthogonal projection onto $kp + ktp$. Moreover, $\overline{y}P_\delta \subseteq P_\delta$ and $\overline{y}P_{\delta^\sigma} \subseteq P_{\delta^\sigma}$. So \overline{y} is a σ -invariant primitive idempotent in $(E(P_\delta) \times E(P_{\delta^\sigma}))^{(t)}$.

Let $T \geq V$ such that $T/V = \langle \theta(t) \rangle$. Then T/V is a complement to N in N^* , as $N_G^*(V, Z) = N_G(V, Z)T$ and $N_G(V, Z) \cap T = V$. Now by Lemma 5.11(ii) the restriction $\pi_{\delta, \delta^\sigma} : E_T(M) \rightarrow (E(P_\Delta) \times E(P_{\Delta^\sigma}))^{(t)}$ is surjective. So by Lemma 2.3(iii) there is a σ -invariant primitive idempotent $y \in E_T(M)$ such that $\pi_{\delta, \delta^\sigma}(y) = \bar{y}$. Then yM is a B -direct summand of $\text{Res}_T^G M$ which lies over Z and Z^* . But $|T/V| = 2$. So $yM \cong \text{Ind}_V^T Z$.

Let B_0 be any symplectic T -form on $\text{Ind}_V^T Z$. Identifying $\text{Ind}_T^G(\text{Ind}_V^T Z)$ with $\text{Ind}_V^G Z$, we regard B_0^G as a symplectic G -form on $\text{Ind}_V^G Z$. Let σ_0 be the adjoint of B_0^G on $E(\text{Ind}_V^G Z)$. Then $(E(P_\Delta) \times E(P_{\Delta^\sigma}), \sigma_0)$ is an involutory N^* -algebra satisfying the hypothesis of Lemma 3.1(ii). So σ_0 is the adjoint of a symplectic N^* -form B_{σ_0} on $P_\Delta + P_{\Delta^\sigma}$ such that $(P_\Delta)^\perp = P_\Delta$ and $(P_{\Delta^\sigma})^\perp = P_{\Delta^\sigma}$.

Let $e \in E_T(\text{Ind}_V^G Z)$ be orthogonal projection onto $1 \otimes \text{Ind}_V^T Z$. Then $\text{tr}_T^G(e) = 1_{\text{Ind}_V^G Z}$ and $e \in \text{tr}_V^T(E_V(\text{Ind}_V^G Z) \Delta E_V(\text{Ind}_V^G Z))$. Set $\bar{e} = \pi_{\Delta, \Delta^\sigma}(e)$, a primitive idempotent in $(E(P_\Delta) \times E(P_{\Delta^\sigma}))^{(t)}$. As O acts trivially on $E(P_\Delta) \times E(P_{\Delta^\sigma})$, we have

$$\begin{aligned} 1_{P_\Delta + P_{\Delta^\sigma}} &= \pi_{\Delta, \Delta^\sigma} \text{tr}_T^G(e) = \text{tr}_{\langle \theta(t) \rangle}^{N^*}(\bar{e}), \quad \text{by Lemma 5.11(ii)} \\ &= \text{tr}_1^N(\bar{e}), \quad \text{as } N^* = N : \langle \theta(t) \rangle. \\ &= \text{tr}_O^{C_H(O)}(\bar{e}) = \text{tr}_1^{C_H(O)}(\bar{e}), \quad \text{as } |O| \text{ is odd.} \\ &= \text{tr}_t^H(\bar{e}), \quad \text{as } H = C_H(O) : \langle t \rangle. \end{aligned}$$

Now $(\bar{e}(P_\Delta + P_{\Delta^\sigma}), B_{\sigma_0}) \cong (k\langle t \rangle, B_t)$ as symmetric $k\langle t \rangle$ -modules. It then follows from Proposition 4.4 that $(P_\Delta + P_{\Delta^\sigma}, B_{\sigma_0}) \cong (kHe_\eta, B_t)$.

As B_t is non-degenerate on $P_\delta + P_{\delta^\sigma}$, we may choose $\bar{\omega}$ so that σ_0 is non-degenerate on $\bar{\omega}(P_\Delta + P_{\Delta^\sigma})$. Then $\bar{\omega}$ belongs to $E(P_\Delta) \times E(P_{\Delta^\sigma})$, which is a σ_0 -invariant semi-simple subalgebra of $E(P_\Delta + P_{\Delta^\sigma})$. So Lemma 3.7 implies that the orthogonal projection $\bar{\omega}_1$ onto $\bar{\omega}(P_\Delta + P_{\Delta^\sigma})$ belongs to $(E(P_\Delta) \times E(P_{\Delta^\sigma}))^{N^*}$.

Lemma 5.11(ii) states that $\pi_{\Delta, \Delta^\sigma} : E_G(\text{Ind}_V^G Z) \rightarrow (E(P_\Delta) \times E(P_{\Delta^\sigma}))^{N^*}$ is surjective. So by Lemma 2.3(iii) there is a σ_0 -invariant primitive idempotent $\omega_1 \in E_G(\text{Ind}_V^G Z)$ such that $\pi_{\Delta, \Delta^\sigma}(\omega_1) = \bar{\omega}_1$. Then $\omega_1 \text{Ind}_V^G Z$ is a B_0^G -direct summand of $\text{Ind}_V^G Z$ that is isomorphic to M . This completes the proof of (i) and (ii).

(iii) holds as M has a T -projective G -form, but V is not a symmetric vertex of M . \square

5.6. Theorems 1.3 and 1.4. Now let T be a symmetric vertex of M .

Lemma 5.17. *Let B be a T -projective G -form on M and let H be a subgroup of G . Then B is H -projective if and only if $T \leq_G H$.*

Proof. The ‘if’ implication holds by Proposition 4.4. The ‘only if’ holds if T is a Green vertex of M . So we assume from now on that T is not a vertex of M .

Let V be a Green vertex and let Z be a V -source of M . By Propositions 5.15 and 5.16 there are isometries $(\text{Ind}_V^S(Z), B_0) \rightarrow \text{Res}_S^G(M, B)$ and $(M, B_2) \rightarrow \text{Ind}_S^G(\text{Ind}_V^S(Z), B_0)$. Here S is a symmetric vertex of M containing V , B_0 is an S -form on $\text{Ind}_V^S(Z)$ and B_2 is a G -form on M . Now suppose that there is a kG -isometry $(M, B) \rightarrow \text{Ind}_H^G(L, B_1)$ for some symmetric kH -module (L, B_1) .

Composing the 3 isometries of the previous paragraph produces a kG -isometry $(M, B_2) \rightarrow \text{Ind}_S^G \text{Res}_S^G \text{Ind}_H^G(L, B_1)$. Now Lemma 4.1 gives

$$\text{Ind}_S^G \text{Res}_S^G \text{Ind}_H^G(L, B_1) \cong \bigoplus_{g \in S \backslash G/H} \text{Ind}_{S \cap gH}^G \text{Res}_{S \cap gH}^{gH}({}^g L, {}^g B_1).$$

So by Lemma 3.2 there is a kG -isometry

$$(M, B_3) \rightarrow \text{Ind}_{S \cap {}^g H}^G \text{Res}_{S \cap {}^g H}^{{}^g H}({}^g L, {}^g B_1),$$

for some $g \in G$ and some symmetric G -form B_3 on M . So B_3 is $S \cap {}^g H$ -projective. But S is a symmetric vertex of M . It follows that $S \leq {}^g H$.

Choosing $H = T$, the work above shows that $S = {}^g T$. Then taking H to be any subgroup of G , we get $T \leq_G H$. \square

Here is a precise statement and proof of Theorem 1.4:

Lemma 5.18. *Suppose that M is a self-dual irreducible kG -module, with symmetric G -form B . Then the symmetric vertices of M are determined up to G -conjugacy.*

Let $V \leq T$ where V is a Green vertex and T a symmetric vertex of M and let Z be a V -source of M . Then $\text{Ind}_V^T Z$ is a B -component of $\text{Res}_T^G M$. Moreover, if B_0 is any T -form on $\text{Ind}_V^T Z$, then M is a B_0^G -component of $\text{Ind}_T^G(\text{Ind}_V^T Z)$.

Proof. Let S be any symmetric vertex of M . As B is the unique G -form on M , B is both T and S -projective. Then by Theorem 1.3, $T \leq_G S$ and $S \leq_G T$. So $T =_G S$. So there is only one G -conjugacy class of symmetric vertices of M .

The other conclusions now follow from Propositions 5.13, 5.15 and 5.16. \square

Note that in case $T = V$ is a vertex of M , Proposition 5.13 implies that the defect multiplicity module P_δ of $E(M)$ is $P(k_N)$. But P_δ is an irreducible projective kN -module, by a well-known theorem of R. Knörr. This forces $P_\delta = k_N$. So V is a Sylow 2-subgroup of $N_G(V, Z)$.

5.7. Real 2-blocks. We turn our attention to the 2-blocks of G . Let \mathcal{B} be a real block of kG . So \mathcal{B} is a $k(G \times G)$ -direct summand of kG which is $^\circ$ -invariant. As $E_{G \times G}(kG) = Z(kG)$, we have $\mathcal{B} = kGe_{\mathcal{B}}$, where $e_{\mathcal{B}}$ is an $^\circ$ -invariant primitive idempotent in $Z(kG)$. Let $\omega_{\mathcal{B}}$ be the central character of \mathcal{B} ; the unique k -algebra map $Z(kG) \rightarrow k$ such that $\omega_{\mathcal{B}}(e_{\mathcal{B}}) = 1_k$. Recall that the conjugacy class sums form a basis for the centre $Z(kG)$ of kG . We will use the notation $X^+ := \sum_{x \in X} x$ for the sum of the elements of a subset X of G , taken in kG .

There are 2-regular conjugacy classes C_1, \dots, C_n of G such that $e_{\mathcal{B}} = \sum_{i=1}^n \alpha_i C_i^+$, with non-zero $\alpha_i \in k$. For each i let \bar{i} be the index of the class of inverses of the elements of C_i . Then $\alpha_{\bar{i}} = \alpha_i$, as \mathcal{B} is real. Choose $c_i \in C_i$, such that $c_{\bar{i}} = c_i^{-1}$ if $C_{\bar{i}} \neq C_i$. Set $C_G(c_i)$ and $C_G^*(c_i) = N_G(\{c_i, c_i^{-1}\})$ as the centralizer and extended centralizer of c_i in G , respectively.

Let E_i be a Sylow 2-subgroup of $C_G^*(c_i)$ and let $D_i = C_{E_i}(c_i)$. So $[E_i : D_i] \leq 2$, with equality if and only if C_i is a non-trivial real class. We call D_i a defect group of C_i and E_i an extended defect group of C_i . When C_i and C_j are real classes, we write $(D_i, E_i) \leq_G (D_j, E_j)$ if $D_i^g \leq D_j$ and $E_j = D_j E_i^g$, for some $g \in G$.

R. Brauer showed that $\alpha_j \neq 0_k$ and $\omega_{\mathcal{B}}(C_j^+) \neq 0_K$, for some j . Any such C_j is a defect class of \mathcal{B} . Then D_j is independent of the choice of C_j , and is called a defect group of \mathcal{B} . Moreover $D_i \leq_G D_j$, for all C_i .

R. Gow showed that there is a real defect class C_j . Then E_j is independent of the choice of real C_j [6, Theorem 2.1]. Gow called E_j an extended defect group of \mathcal{B} . In [21, Proposition 14] we proved that $(D_i, E_i) \leq_G (D_j, E_j)$, for all real C_i . In particular, we can and do choose c_i so that $D_i \leq D_j$, and moreover $E_j = D_j E_i$, if C_i is real. So for real C_i we have $c_i^{e_i} = c_i^{-1}$ for some $e_i \in E_j \setminus D_j$.

Now consider the involutory G -algebra $(kG, {}^o)$. Each subgroup H of G defines a trace map $\text{tr}_H^G : kG^H \rightarrow Z(kG)$. Then for all i

$$C_i^+ = \text{tr}_{C_G(c_i)}^G(c_i) = \text{tr}_{D_i}^G(c_i), \quad \text{as } [C_G(c_i) : D_i] \text{ is odd.}$$

Set $\theta = \sum_{i=1}^n \alpha_i \text{tr}_{D_i}^{E_j}(c_i)$. As $C_i^+ = \text{tr}_{E_j}^G(\text{tr}_{D_i}^{E_j}(c_i))$ and $\text{tr}_{D_i}^{E_j}(c_i^o) = \text{tr}_{D_i}^{E_j}(c_i)^o$ and $\alpha_i^o = \alpha_i$, we have

$$\theta \in kG^{E_j}, \quad \theta^o = \theta \quad \text{and} \quad e_{\mathcal{B}} = \text{tr}_{E_j}^G(\theta).$$

We use this to prove parts (i) and (ii) of Theorem 1.5:

Lemma 5.19. *Each symmetric vertex of an indecomposable \mathcal{B} -module is contained in an extended defect group of \mathcal{B} .*

Proof. Let (M, B) be a symmetric \mathcal{B} -module, where M is indecomposable, and let σ be the adjoint of B on $E(M)$. Let $\pi : kG \rightarrow E(M)$ define the module structure on M . Then π maps kG^H into $E_H(M)$, for all $H \leq G$. Moreover $\pi(g)^\sigma = \pi(g^{-1})$, for all $g \in G$, as B is G -invariant. So $\pi(\theta)$ is a σ -invariant element of $E_{E_j}(M)$ and

$$\text{tr}_{E_j}^G(\pi(\theta)) = \pi(\text{tr}_{E_j}^G(\theta)) = \pi(e_{\mathcal{B}}) = 1_M.$$

We conclude from Proposition 4.4 that B is E_j -projective. \square

Corollary 5.20. *There is a self-dual irreducible \mathcal{B} -module whose symmetric vertices are the extended defect groups of \mathcal{B} .*

Proof. If \mathcal{B} is the principal 2-block of G , the symmetric vertices of the trivial kG -module are the Sylow 2-subgroups of G . These are also the extended defect groups of \mathcal{B} . Suppose then that \mathcal{B} is not the principal block. Now [8, Proposition 1.4(v)] implies that the defect group D_j of \mathcal{B} is a Green vertex of some self-dual irreducible \mathcal{B} -module M . Let T be a symmetric vertex of M containing D_j . Then $[T : D_j] = 2$, by Theorem 1.2. But $T \leq_G E_j$, according to Lemma 5.19. Since $[E_j : D_j] = 2$ we conclude that $T =_G E_j$ is an extended defect group of \mathcal{B} . \square

To prove part (iii) of Theorem 1.5, we work in the involutory $G \times G$ -algebra $(E(kG), \sigma)$. Here σ is the adjoint of B_1 , as in Section 5.1. Recall the identification (4) of $x \otimes y$ as an endomorphism of kG , for $x, y \in kG$. Then $(g_1, g_2) \cdot x \otimes y = g_1 x g_2^{-1} \otimes g_1 y g_2^{-1}$, for all $g_1, g_2 \in G$, as can be checked. So

$$\text{Stab}_{G \times G}(x \otimes y) = \{(g_1, g_2) \in G \times G \mid g_1 x = x g_2, g_1 y = y g_2\}.$$

Now $C_G(x) = C_G(x^2)$, if $x \in G$ has odd order. It follows from this that

$$\text{Stab}_{G \times G}(x \otimes x^{-1}) = \Delta C_G(x), \quad \text{if } x \in G \text{ is 2-regular.}$$

Set $\Delta H = \{(h, h) \mid h \in H\}$, for $H \leq G$. Let D be a defect group of x and let C be the conjugacy class of G containing x^2 . Then $\text{tr}_{\Delta D}^{G \times G}(x \otimes x^{-1}) = C^+$.

Recall that C_j is a real defect class of \mathcal{B} , D_j is a defect group of \mathcal{B} and E_j is an extended defect group of \mathcal{B} . Moreover each class C_i occurring in $e_{\mathcal{B}}$ is 2-regular. So there is a unique $d_i \in \langle c_i \rangle$ such that $d_i^2 = c_i$. Define

$$\Theta = \sum_{i=1}^n \alpha_i \text{tr}_{\Delta D_i}^{\Delta E_j}(d_i \otimes d_i^{-1}).$$

If c_i is real, then $c_i^{e_i} = c_i^{-1}$, where $e_i \in E_j \setminus D_j$. So $e_i d_i e_i^{-1} = d_i^{-1}$ and

$$(e_i, e_i) \cdot d_i \otimes d_i^{-1} = d_i^{-1} \otimes d_i = (d_i \otimes d_i^{-1})^\sigma, \quad \text{using (5).}$$

It follows from this that $\text{tr}_{\Delta D_i}^{\Delta E_j}(d_i \otimes d_i^{-1}) = \text{tr}_{\Delta D_i}^{\Delta E_j}(d_i \otimes d_i^{-1})^\sigma$, for all i . So

$$(7) \quad \Theta^\sigma = \Theta \quad \text{and} \quad \text{tr}_{\Delta E_j}^{G \times G}(\Theta) = e_{\mathcal{B}}.$$

Now notice that $e_{\mathcal{B}}$ is the identity element of $E_{G \times G}(\mathcal{B}) = Z(\mathcal{B})$.

Lemma 5.21. *Let E be an extended defect group of \mathcal{B} . Then the restriction of B_1 to \mathcal{B} is non-degenerate and ΔE -projective. So ΔE is a symmetric vertex of \mathcal{B} .*

Proof. Let $D \leq E$ be a defect group of \mathcal{B} . So ΔD is a Green vertex of \mathcal{B} as $k(G \times G)$ -module. Now \mathcal{B} is the only direct summand of kG isomorphic to its $k(G \times G)$ -dual \mathcal{B}^* . So B_1 is non-degenerate on \mathcal{B} , by Lemma 3.8. Then (7) and Proposition 4.4 imply that B_1 is ΔE -projective.

Suppose that \mathcal{B} is the principal 2-block of G . Then $E = D$ is a defect group of \mathcal{B} . So ΔE is, as $k(G \times G)$ -module. As B_1 is ΔE -projective, Proposition 5.13 implies that ΔE is a symmetric vertex of \mathcal{B} .

Suppose that \mathcal{B} is not the principal 2-block of G . Then \mathcal{B} belongs to a non-principal 2-block of $G \times G$. So ΔD is not a symmetric vertex of \mathcal{B} , by Corollary 5.14. As B_1 is ΔE -projective and $[\Delta E : \Delta D] = 2$, Proposition 5.15 implies that ΔE is a symmetric vertex of \mathcal{B} . \square

5.8. Examples. We give examples to show that all cases in Theorem 1.2 actually occur. So as above let M be a self-dual indecomposable kG -module of symmetric type, let T be a symmetric vertex of M , let V be a Green-vertex of M contained in T and let Z be a V -source of M . In view of the literature on vertices, most of our examples involve a self-dual irreducible kG -module.

To get examples of Theorem 1.2(i), choose M so that V is a Sylow 2-subgroup of G . Then the symmetric vertex T will also be a Sylow 2-subgroup of G . For a non-trivial example, if S_{2n} is the symmetric group of degree $2n \geq 6$ and M is the irreducible kS_{2n} -module labelled by the 2-regular partition $[2n-1, 1]$, then V is a Sylow 2-subgroup of S_{2n} according to [20].

There are examples of Theorem 1.2(i) where V is not a Sylow 2-subgroup of G : let M be the 4-dimensional irreducible kS_5 -module labelled by $[3, 2]$. Then according to [3] V is conjugate to a Klein-four subgroup of $A_4 \leq S_5$ and $\text{Res}_V^{S_5}(M) = Z_1 \oplus Z_2$ is the direct sum of two non-isomorphic self-dual V -sources. Then it follows from Lemma 3.8 and Proposition 5.13 that $V = T$ is a symmetric vertex of M .

Now suppose that G has even order and M is irreducible and projective as kG -module. Then M belongs to a real 2-block \mathcal{B} of G which has a trivial defect group $V = 1$. According to [6], there is an involution $t \in G$ such that $\langle t \rangle$ is an extended defect group of \mathcal{B} . It then follows from Lemma 4.6 that $\langle t \rangle$ is a symmetric vertex of M . As $Z = k$ is self-dual, this gives an example of Theorem 1.2(ii).

Next let M be the irreducible kS_7 -module labelled by $[4, 3]$. It is shown in [3] that M shares its vertices and sources with the irreducible kS_5 -module labelled by $[3, 2]$, as discussed above. Once again we have an example of Theorem 1.2(ii), but here the vertex V is not a defect group of the 2-block \mathcal{B} .

It is relatively difficult to find examples of Theorem 1.2(iii). Take $n \geq 3$, and set $H = \text{GL}(n, 2)$ and $G = H : \langle \tau \rangle$, where τ is the transpose inverse automorphism of H . Let M_1 be the natural n -dimensional kH -module and set $M = \text{Ind}_H^G M_1$. Then V is a Sylow 2-subgroup of H and $Z = \text{Res}_V^H(M_1)$ is a V -source of M . It is not too difficult to check that Z is not self-dual, giving us an infinite family of examples of

Theorem 1.2(iii). We note that in case $n = 4$, $H \cong A_8$ and $G \cong S_8$. Then we can identify M with the irreducible kS_8 -module labelled by $[5, 3]$.

E. C. Dade [2] gave us an example of a solvable group which has a self-dual irreducible module whose sources are not self-dual: Let G be the semi-direct product of an extra-special group of order 27 and exponent 3 with a semi-dihedral group of order 16. Take M to be the unique 6-dimensional kG -module. Then V is a quaternion group of order 8 and $\text{Res}_V^G M = Z \oplus Z^*$ with $Z \not\cong Z^*$. We describe this in more detail in [23].

In [23] the author and G. Navarro use the results of the current paper to show that if G is solvable, and M is irreducible, then case (i) of Theorem 1.2 does not occur. Moreover a symmetric vertex of M splits over a Green vertex if and only if $P(M)$ is of quadratic type, in the sense of [8].

6. ACKNOWLEDGEMENT

Many years ago Burkhard Külshammer challenged me to generalise the notion of extended defect groups from real 2-blocks to other self-dual objects. This inspired me to write this paper. Discussions with Adam Allan, Everett C. Dade, Rod Gow and Jürgen Müller helped to clarify my ideas. I would also like to thank the anonymous referees for picking up a number of errors, in particular a false conclusion in an earlier version of Lemma 2.3.

REFERENCES

- [1] R. Brauer, ‘Zur Darstellungstheorie der Gruppen endlicher Ordnung’, *Math. Zeit.* **63** (1956) 406–444.
- [2] E. C. Dade, ‘A self-dual simple module with a non self-dual source’, email July, 2015.
- [3] S. Danz, B. Külshammer, R. Zimmermann, ‘On vertices of simple modules for symmetric groups of small degrees’, *J. Algebra* **320** (2008) 680–707.
- [4] P. Fong, ‘On decomposition numbers of J_1 and $R(q)$ ’, *Symposia Mathematica* Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), pp. 415–422. (Academic Press, London, 1974).
- [5] R. Gow, Real 2-blocks of S_n , A_n and their double covers, *Proc. Symp. Pure Math.* **47** (1987) 421–429.
- [6] R. Gow, ‘Real 2-blocks of characters of finite groups’, *Osaka J. Math.* **25** (1) (1988) 135–147.
- [7] R. Gow, T. J. Laffey, ‘Pairs of alternating forms and products of two skew symmetric matrices’, *Linear Alg. and Applic.* **63** (1984) 119–132.
- [8] R. Gow and W. Willems, ‘Quadratic geometries, projective modules and idempotents’, *J. Algebra* **160** (1993) 257–272.
- [9] R. Gow and W. Willems, ‘A note on Green correspondence and forms’, *Comm. Algebra* **23** (4) (1995) 1239–1248.
- [10] J. A. Green, ‘On the indecomposable representations of a finite group’, *Math. Zeit.* **70** (1959) 430–445.
- [11] J. A. Green, ‘Blocks of modular representations’, *Math. Zeit.* **79** (1962) 100–115.
- [12] L. C. Grove, *Classical Groups and Geometric Algebra*, Graduate Studies in Math. 39, (Amer. Math. Soc. 2002).
- [13] D. G. Higman, ‘Modules with a group of operators’, *Duke Math. J.* **21** (1954) 369–376.
- [14] G. D. James, *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math. 682 (Springer-Verlag, Berlin Heidelberg New York 1978).
- [15] B. Külshammer, *Lectures on Block Theory*, Lond. Math. Soc. Lecture Notes 161, (Cambridge Univ. Press 1991).
- [16] B. Külshammer, ‘Roots of simple modules’, *Canad. Math. Bull.* **49** (1) (2006) 96–107.
- [17] B. Külshammer, R. Zimmermann, ‘On a question of Willems’, preprint 2003, 2 pages.
- [18] P. Landrock and O. Manz, ‘Symmetric forms, idempotents and involutory anti-isomorphisms’, *Nagoya Math. J.* **125** (1992) 33–51.

- [19] T. G. Lee, ‘Group identities on units and symmetric units of group rings’, *Algebra and Applications*, 12. Springer-Verlag London, Ltd., London, 2010.
- [20] J. Müller, R. Zimmermann, Green vertices and sources of simple modules of the symmetric group labelled by hook partitions, *Arch. Math. (Basel)* **89** (2) (2007) 97–108.
- [21] J. Murray, ‘Strongly real 2-blocks and the Frobenius-Schur indicator’, *Osaka J. Math.* **43** (1) (2006) 201–213.
- [22] J. Murray, ‘Projective indecomposable modules, Scott modules and the Frobenius-Schur indicator’, *J. Algebra* **311** (2007) 800–816.
- [23] J. C. Murray, G. Navarro, ‘Characters, bilinear forms and solvable groups’, *J. Algebra* **449** (2016) 346–354. <http://dx.doi.org/10.1016/j.jalgebra.2015.10.024>.
- [24] H. Nagao and Y. Tsushima, *Representations of finite groups*, (Academic Press, 1987).
- [25] G. Nebe, ‘Orthogonal Frobenius reciprocity’, *J. Algebra* **225** (2000) 250–260.
- [26] E. Pacifici, ‘On tensor induction for representations of finite groups’, *J. Algebra* **288** (2) (2005) 287–320.
- [27] G. R. Robinson, ‘The Frobenius-Schur Indicator and Projective Modules’, *J. Algebra* **126** (1989) 252–257.
- [28] J. Thévenaz, *G-algebras and Modular Representation Theory*, Oxford Science Publications, (Clarendon Press, Oxford, 1995).
- [29] John G. Thompson, Bilinear forms in characteristic p and the Frobenius-Schur indicator, Group Theory, Beijing 1984, *Lecture Notes in Math.* **1185**, Springer-Verlag (1984) 210–230.
- [30] W. Willems, ‘Metrische Moduln über Gruppenringen’, Ph.D. Thesis, Johannes Gutenberg-Universität, Mainz, 1976.

MAYNOOTH UNIVERSITY, DEPARTMENT OF MATHEMATICS & STATISTICS, NATIONAL UNIVERSITY OF IRELAND, MAYNOOTH, CO. KILDARE, IRELAND

E-mail address: `John.Murray@maths.nuim.ie`